INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 3, Pages 348–360

A SYMMETRIC FINITE VOLUME ELEMENT SCHEME ON QUADRILATERAL GRIDS AND SUPERCONVERGENCE

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(Communicated by Zhimin Zhang)

Abstract. A symmetric finite volume element scheme on quadrilateral grids is established for a class of elliptic problems. The asymptotic error expansion of finite volume element approximation is obtained under rectangle grids, which in turn yields the error estimates and superconvergence of the averaged derivatives. Numerical examples confirm our theoretical analysis.

Key Words. quadrilateral grids, symmetric finite volume schemes, asymptotic error expansion, superconvergence.

1. Introduction

Finite volume methods are a class of important numerical methods to solve PDEs ([2, 6, 8, 9, 14]), which can be viewed as a bridge between finite element methods and finite difference methods. Due to being able to preserve some physical conservation properties locally, such as mass, momentum and energy conservation, finite volume methods are widely applied in many fields, such as computational fluid dynamics and computational physics and so on.

The standard finite volume discretizations usually generate a linear systems with asymmetric matrix for self-adjoint elliptic problems, in many cases, the symmetry is the fundamental physical principle of reciprocity. This asymmetry leads to the fact that many efficient iterative methods which are suitable for solving the symmetric linear systems, such as the conjugate gradient method, can't be employed. It is interesting to see if there exist finite volume schemes that are symmetry preserving. Recently, Aihui Zhou and Xiuling Ma([10, 11, 13]) proposed a class of symmetric finite volume schemes under the triangular grids for solving the self adjoint elliptic boundary value problems and parabolic problems, which gave a positive answer for the triangular grids. However, the answer is still open for the quadrilateral grids so far. For quadrilateral grids, the non-constant derivatives of finite volume element makes the analysis more difficult since there is no convenient weak form.

In this paper, by choosing vertex-centered type control volume properly and using finite volume element methods to discretize the balance equation, a symmetrical finite volume scheme on quadrilateral grids is established. Different from the symmetrical finite volume scheme on the triangular grid, There is no weak form available, so the convergence analysis is more difficult. Here we give a detailed analysis for rectangle grids only. The main ingredients are the bound estimate of

Received by the editors August 1, 2004 and, in revised form, March 22, 2005.
2000 *Mathematics Subject Classification.* 65N30,65F10.
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Subsidized by NASF (10376031),Research Fund for Doctoral Program of Chinese Ministry of Education and by National High-Tech ICF Committee in China.

the minimum eigenvalue for the coefficient matrix of our scheme and asymptotic expansions of the truncation error. The asymptotic error expansion of finite volume element approximation is obtained under rectangle grids, which in turn yields the error estimates and superconvergence of the averaged derivatives. Numerical examples confirm our theoretical analysis and show the efficiency of the method on general quadrilateral grids.

2. Preliminary

In this paper, we consider the following model problem,

$$\begin{cases} -\nabla(a(x)\nabla u) &= f, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \in R^2$ is a convex polygonal domain with boundary $\partial\Omega$, $x = (x_1, x_2), c_1 \le a(x) \le c_2$ and c_1, c_2 are two positive real numbers.

For simplicity, we introduce the notation $\lesssim, \gtrsim$ as same as that in paper ([3]) which means that when we write $A \lesssim B, A \gtrsim B$ then there exist two positive constant c and C such that $A \le cB, A \ge CB$ respectively.

Let $\mathcal{P}_{1,1} = \{a_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_1\xi_2 : \ a_l \in \mathbf{R}, l = 0(1)3\}$ be the set of bilinear polynomial, and $W^{m,p}(\Omega)$ be the Sobolev space with the norms:

$$\|v\|_{m,p} = \Big(\sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}\Big)^{\frac{1}{p}}, \ 1 \le p < \infty,$$
$$\|v\|_{m,\infty} = \max_{|\alpha|\le m} \|D^\alpha v\|_{L^\infty(\Omega)}, \ p = \infty,$$

where $\alpha = (\alpha_1, \alpha_1, ..., \alpha_n)$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $\alpha_i > 0, \ 1 \le i \le n$.

In addition, we assume that $\Omega^h = \{E_i, \ 1 \le i \le M\}$ is any given quadrilateral grid of Ω (shown as Fig. 1(a) and (b)), and $X = \{X_i = (x_1^i, x_2^i), \ 1 \le i \le N\}$ is the set of all nodes in Ω^h, where M and N are the total numbers of all partition elements and nodes respectively.

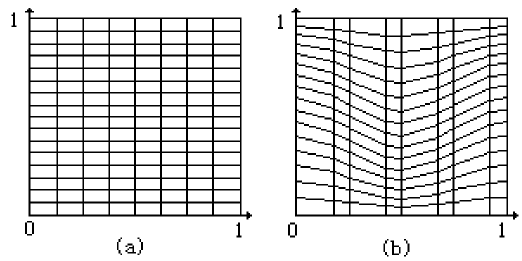

FIGURE 1. (a) uniform grids. (b) non-orthogonal grids.

In order to establish the finite volume scheme, we need to introduce the dual partition $\Omega_*^h = \{b_{X_i}, \ 1 \le i \le N\}$ of Ω^h, where b_{X_i} be the dual element(control volume) of the node X_i shown in Fig. 2(a). In this figure, $O_{i_l}, 1 \le l \le 4$ is the "center" of the l-th quadrilateral element neighboring to X_i, which is mapped from the center of the reference unit square element E shown as in Fig. 2(b) by the bilinear isoparametric transformation , and $M_{i_l}, 1 \le l \le 4$ are midpoints of all edges connected with X_i. Additionally, for any quadrilateral element E_k, we call