INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 3, Pages 371–376

SOME NEW LOCAL ERROR ESTIMATES IN NEGATIVE NORMS WITH AN APPLICATION TO LOCAL A POSTERIORI ERROR ESTIMATION

ALFRED H. SCHATZ

Abstract. Here we survey some previously published results and announce some that have been newly obtained. We first review some of the results in [3] on estimates for the finite element error at a point. These estimates and analogous ones in [4] and [7] have been applied to problems in a posteriori estimates [2], [8], superconvergence [5] and others [9], [10]. We then discuss the extension of these estimates to local estimates in L_∞ based negative norms. These estimates have been newly obtained and are applied to the problem of obtaining an asymptotically exact a posteriori estimator for the maximum norm of the solution error on each element.

Key Words. Superconvergence, error estimate, a posteriori

1. Introduction

This is a survey paper whose aim is threefold. First we will discuss some of the results in [3]. There various error estimates for the finite element method for second order elliptic problems were derived. In particular, sharp error estimates for the solution and gradient at a point were given which more clearly showed the dependence of the error at the point on the solution at, and also away from the point. Some very useful inequalities for applications are so-called "asymptotic expansion inequalities". These are simple consequences of the error estimates and are the key in [2] and [8] on local a posteriori estimators, in [6] on superconvergence, in [9] on asymptotic expansions, and in [10] on Richardson extrapolation.

Our second aim is to present some newly obtained error estimates that are extensions of the estimates discussed above together with associated asymptotic expansion inequalities.

Finally our third aim is to apply these new asymptotic expansion inequalities to obtain an asymptotically exact a posteriori estimator for the maximum norm of the solution on each element. Some asymptotically exact estimators for the maximum norm of the gradient on each element that were given in [2] and [8] are very closely related.

This paper is organized as follows. Section 1(a) contains some preliminaries both for the Neumann problem we shall discuss and the finite element method we shall use. Section 1(b) contains special cases of the error estimates from [3]. Section 2 contains some new estimates. Namely, we give error estimates in local negative norms (L_∞ based) and corresponding asymptotic error expansion inequalities. Section 3 contains a discussion of the application of the results of Section 2 to a problem in a posteriori estimation.

Received by the editors August 1, 2004.
2000 *Mathematics Subject Classification.* 65C30.

I would like to thank the organizers of the conference "Superconvergence and A Posteriori Error Estimates" at Hunan University for an excellent meeting and their warm hospitality.

(a) Preliminaries for a smooth Neumann problem

Let $\Omega \subset\subset R^N$ be a domain with a smooth boundary $\partial\Omega$. Consider the boundary value problem

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f \text{ in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n_L} = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $\frac{\partial u}{\partial n_L}$ is the conormal derivative. The weak formulation of (1.1), (1.2) for $u \in W_2^1(\Omega)$ is

$$\begin{aligned} A(u,v) &= \int_\Omega \Big(\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i}v + c(x)uv\Big)dx \\ &= \int_\Omega fvdx \text{for all} v \in W_2^1(\Omega). \end{aligned} \tag{1.3}$$

We assume that the coefficients are smooth and $A(\cdot,\cdot)$ is uniformly elliptic and coercive, i.e., there exist constants $C_{\text{ell}} > 0$, $C_{\text{co}} > 0$ such that

$$\begin{aligned} \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j &\geq C_{\text{ell}}|\xi|^2, \text{ for all } \xi \in R^N \\ C_{\text{co}}\|v\|^2_{W_2^1(\Omega)} &\leq A(v,v) \text{ for all } v \in W_2^1(\Omega). \end{aligned} \tag{1.4}$$

It is well known that a unique solution of (1.3) exists for each $f \in (W_2^1(\Omega))'$, and if f is smooth then so is u.

Consider the approximation of u using the finite element method. To this end let $0 < h < 1$ be a parameter, $r \geq 2$ an integer, and $S_r^h(\Omega) \subset W_\infty^1(\Omega)$, be a family of finite elements. The precise assumptions on the finite element spaces will not be given here (see [3]) but roughly speaking they are satisfied by many types of commonly used finite elements. For the purpose of this presentation we take them to be any one of a variety of continuous functions, whose restriction to each set τ of a quasi–uniform partition (of roughly size h) that fits the boundary exactly and contains all polynomials of degree $r-1$ ($r=2$ piecewise linear, $r=3$ piecewise quadratic, etc.). Thus they can approximate functions to order h^r in L_∞ and order h^{r-1} in $W_\infty^1(\Omega)$. We now define $u_h \in S_r^h$, the finite element approximation to u, as the solution of

$$A(u_h,\varphi) = \int_\Omega f\varphi dx, \text{ for all } \varphi \in S_r^h(\Omega), \tag{1.5}$$

or

$$A(u-u_h,\varphi) = 0, \text{ for all } \varphi \in S_r^h(\Omega). \tag{1.6}$$

(b) Some known pointwise error estimates

Define, for $d > 0$ and fixed $x \in \overline{\Omega}$

$$B_d(x) = \{y \in \Omega : |x-y| < d\}. \tag{1.7}$$

The following are special cases of error estimates given in [3].