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ON A FINITE DIFFERENCE SCHEME FOR A BEELER-REUTER BASED MODEL OF CARDIAC ELECTRICAL ACTIVITY

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Abstract. We investigate an explicit finite difference scheme for a Beeler-Reuter based model of cardiac electrical activity. As our main result, we prove that the finite difference solutions are bounded in the L^∞-norm. We also prove the existence of a weak solution by showing convergence to the solutions of the underlying model as the discretization parameters tend to zero. The convergence proof is based on the compactness method.

Key Words. reaction-diffusion system of Beeler-Reuter type, excitable cells, cardiac electric field, monodomain model, finite difference scheme, maximum principle, convergence

1. Introduction

The purpose of this paper is to study a finite difference scheme for a mathematical model that describes electrical activity in cardiac tissue. A spatially dependent model for this phenomenon is commonly written

$$\begin{aligned} \frac{\partial v}{\partial t} &= \nabla \cdot (M \nabla v) - I_{ion}(s, v), \\ \frac{ds}{dt} &= F(s, v), \end{aligned} \tag{1}$$

where v is the transmembrane potential, M is the (diagonal) conductivity tensor, and s is a state vector whose entries depend on the cell model. This reaction-diffusion system is commonly referred to as the monodomain model of electrophysiology, and the complexity depends on the cell model represented by the ODE system. A more general model (which is not treated here) for electrical activity in anisotropic cardiac tissue is the so-called bidomain model. In this model the cardiac muscle is viewed as a superposition of two (anisotropic) continuous media, referred to as the intracellular and the extracellular. The intracellular and extracellular media are connected by a continuous cellular membrane, and this coupling gives rise to a reaction-diffusion system of degenerate type [5], where the unknowns are the intracellular potential u_i, the extracellular potential u_e, and the transmembrane potential $v = u_i - u_e$ (i.e., the jump in the potential across the cellular membrane). Under the assumption of equal anisotropies, (that is, the ratio of the conductivity coefficients parallel and transverse to the direction of fibre is constant, both for

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the intracellular and extracellular media), the bidomain model reduces to the monodomaim model (1). In this case, the intracellular and extracellular potential u_i and u_e can be recovered by scaling the transmembrane potential v appropriately.

The cell model represented by the system of ODEs in (1) could be rather simple, as in, e.g., the well known FitzHugh-Nagumo model, or, as in recent models, very complex, see, e.g., Winslow *et al.* [10], where s contains dozens of variables, such as membrane channels and ionic concentrations. In the present paper, we investigate the monodomain model coupled to the Beeler-Reuter equations [1], which was one of the first mathematical models to be developed for describing the electrophysiology of a cardiac cell. Compared to more recent models, this is a simple yet fairly realistic description of cell dynamics due to the presence of the intracellular calcium concentration, which is important for contraction of the heart. Physiological, as well as mathematical, considerations impose certain constraints or bounds on the calcium concentration, and it is our aim here to identify actual values for these bounds by analysing a finite difference scheme. We point out that the upper bound on the calcium concentration found in this study depends greatly on the governing equation of this quantity, and does not necessarily have a significant physiological interpretation. More advanced models for describing intracellular calcium dynamics have been developed, by, e.g, Luo and Rudy [8], Noble [2], and Winslow *et al.* [10].

During the depolarization phase of the heart, the solution is rapidly changing, i.e., there are steep solution gradients present. Thus there is a need for a strict requirement on the time step in any numerical scheme that wishes to resolve this feature of the solution. We identify such a constraint when proving a maximum principle for an explicit finite difference scheme approximating the system (1) with Beeler-Reuter kinetics, which is our main result in this paper. In addition, we give a rather simple convergence proof for the finite difference scheme for a large class of initial values for the transmembrane potential v. We prove L^2 convergence of the finite difference solutions using the compactness method and the concept of weak solution. In passing, we mention that due to the sharp transition layers in the solution, it is reasonable to seek weak solutions of the system. For convergence of some numerical schemes for (1) with the simpler Hodgkin-Huxley type kinetics, see [9, 6] and the references cited therein.

The remainder of this paper is organised as follows: In Section 2 we present the mathematical model. Section 3 is devoted to an informal motivation at the continuous level of why lower and upper bounds should be available for the solutions of the reaction-diffusion system (1) with Beeler-Reuter kinetics. We give a weak formulation of the problem in Section 4, while the finite difference scheme is presented in Section 5. In Section 6 we prove the maximum principle for the scheme, while the convergence of the scheme is proved in Section 7. We conclude the paper by showing some simulation results in Section 8.

Throughout this paper we denote a generic constant that does not depend on the discretization parameters by K. The actual value of K may change from one line to the next during a computation.

2. Model description

In this section we present the mathematical model to be studied. We confine the discussion to two spatial dimensions, but the extension to higher dimensions is straightforward. We consider a bounded open domain $\Omega \subset \mathbf{R}^2$, a fixed final time