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AN AXIOMATIC APPROACH TO NUMERICAL APPROXIMATIONS OF STOCHASTIC PROCESSES

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This paper is dedicated to emeritus professor P. Heinz Müller at TU Dresden.

Abstract. An axiomatic approach to the numerical approximation Y of some stochastic process X with values on a separable Hilbert space H is presented by means of Lyapunov-type control functions V. The processes X and Y are interpreted as flows of stochastic differential and difference equations, respectively. The main result is the proof of some extensions of well-known deterministic principle of Kantorovich-Lax-Richtmeyer to approximate solutions of initial value differential problems to the stochastic case. The concepts of invariance, smoothness of martingale parts, consistency, stability, and contractivity of stochastic processes are uniquely combined to derive efficient convergence rates on finite and infinite time-intervals. The applicability of our results is explained with drift-implicit backward Euler methods applied to ordinary stochastic differential equations (SDEs) driven by standard Wiener processes on Euclidean spaces $H = \mathbb{R}^d$ along functions such as $V(x) = \sum_{i=0}^{k} c_i x^{2i}$. A detailed discussion on an example with cubic nonlinearity from field theory in physics (stochastic Ginzburg-Landau equation) illustrates the suggested axiomatic approach.

Key Words. stochastic differential equations, numerical methods, stochastic difference equations, convergence, stability, contractivity, stochastic Kantorovich-Lax-Richtmeyer principle, Lyapunov-type functions, worst case convergence rates

1. Introduction

Many dynamic problems in Natural Sciences, Engineering, Environmental Sciences and Econometrics lead to models governed by nonlinear and dissipative stochastic ordinary and partial differential systems. These systems are explicitly solvable very rarely. Thus one has to resort to numerical approximations. In deterministic theory there are well-known principles for the approximation of their solutions in appropriate Banach spaces. Two of them are the principles of Kantorovič [17], [11] and Lax and Richtmeyer [24], [34], combining stability, consistency and convergence for well-posed problems. However, in the stochastic case, there is substantially less known about their counterparts. We are going to continue our works exhibited in [35] - [46] by establishing basic approximation principles for stochastic processes X, Y which have values in random Hilbert spaces H or Banach spaces with norms defined via subadditive pseudo-bilinear forms. As the simplest application we bear

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in mind the case of stochastic ordinary differential equations (SDEs) and their numerical approximations with variable step sizes. (An application to some types of stochastic partial differential equations (SPDEs) with appropriate relation between space- and time-discretization for their approximations is conceivable too, but left to future work). In this paper the time-evolution of the global discretization-error is considered without taking into account any discretization of the state space. Note that the herein suggested axiomatic approach to the analysis of numerical approximations is especially efficient within the framework of "eigenfunction approach" applied to quasilinear SPDEs.

For the description of the approximation problem we assume the following. Fix a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ with deterministic finite time-interval $[0,T]$. Let $H = H(\omega)$ be a separable random Hilbert space with $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted scalar product $< .,. >_H$ and real numbers as its scalars, and let μ be any nonrandom, σ-finite, positive measure on $([0,T], \mathcal{B}([0,T]))$. Here $\mathcal{B}(.)$ represents the σ-field of all Borel-sets of the inscribed set. $X = (X_t(\omega))_{0\leq t\leq T}$ and $Y = (Y_t(\omega))_{0\leq t\leq T}$ denote two $(\mathcal{F}_t)$-adapted stochastic processes on the given probability space with values in one and the same Hilbert space H. Then, obviously, the vector space

$$H_2([0,T],\mu,H) := \left\{ X = (X_t(\omega))_{0\leq t\leq T} : \begin{array}{l} X_t(\omega) \in H(\omega) \text{ for all times } t, \\ X_t \text{ is } (\mathcal{F}_t, \mathcal{B}(H)) - \text{measurable}, \\ X \text{ cadlag with respect to time } t, \\ \int_0^T \mathbb{E} < X_t, X_t >_H \, d\mu(t) < +\infty \end{array} \right\}$$

forms a Hilbert space with scalar product

$$< X, X >_{H_2} := \int_0^T \mathbb{E} < X_t, X_t >_H \, d\mu(t)$$

and real numbers as its scalars. The naturally induced norms are given by

$$||X||_H := \sqrt{< X, X >_H}, \quad ||X||_{H_2} := \sqrt{< X, X >_{H_2}}.$$

We are interested to tackle the approximation problem of X by Y (and also Y by X, thanks to the inherent symmetry) on this space, in particular, on the subset

$$\mathbb{D}_T = \left\{ X \in H_2([0,T],\mu,H) : \sup_{0\leq t\leq T} \mathbb{E} < X_t, X_t >_H < +\infty \right\}.$$

Furthermore, let $[K]_- \geq 0$ denote the negative part of K, and $[K]_+ \geq 0$ its positive part such that we have $K = [K]_+ - [K]_-$.

The paper is organized as follows. Section 2 commences with the statement of main concepts and assumptions to prove a fairly general approximation theorem for convergence rates of numerical approximations with variable step sizes. In Sections 3 and 4 we present two versions of this theorem for the most general and dissipative case. The main purpose of this paper is to publish a fairly complete proof of universal error estimates for the approximation of some Hilbert-space-valued stochastic processes while incorporating information on certain Lyapunov-function(al)s $V = V(x)$. This significantly extends the applicability of our original work [45] where we only considered the very restricted case of $V(x) = 1+||x||^2$ from practical point of view (cf. example in Section 6.2). The main theorems 3.1 and 4.1 have already been formulated in [44], but without any detailed proof-steps. Here the complete proof incorporating the role of Lyapunov-functions $V(x)$ (much more general than $V(x) = 1 + ||x||^2$) is presented by dividing it into a series of auxiliary lemmas as done in Section 5. Section 6 briefly discusses the fairly transparent case of ordinary stochastic differential equations and drift-implicit Euler methods in $\mathbb{R}^d$,