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AN ε-UNIFORM FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. This work develops an ε-uniform finite element method for singularly perturbed two-point boundary value problems. A surprising and remarkable observation is illustrated: By inserting one node arbitrarily in any element, the new finite element solution always intersects with the original one at fixed points, and the errors at those points converge at the same rate as regular boundary value problems (without boundary layers). Using this fact, an effective ε-uniform approximation out of boundary layer is proposed by adding one point only in the element that contains the boundary layer. The thickness of the boundary layer need not be known *a priori*. Numerical results are carried out and compared to the Shishkin mesh for demonstration purpose.

Key Words. finite element method, singular perturbation, ε-uniform approximation, layer-adapted mesh, Shishkin mesh.

1. Introduction

This paper is concerned with linear Galerkin finite element method for singularly perturbed boundary value problems (BVPs). Consider a one-dimensional BVP problem

$$-\varepsilon u'' - bu' + cu = f, \quad x \in (0,1); \quad u(0) = u(1) = 0. \tag{1.1}$$

For simplicity, let $b \leq 0$, $c \geq 0$, and $0 < \varepsilon \ll 1$ be constants such that not both b and c are 0. If $b > 0$, by using substitution $w(x) = u(1-x)$, it reduces to the case with $b \leq 0$. All results presented in this paper can be readily generalized to smooth and non-vanishing functions $b(x)$ and $c(x)$.

If the exact solution $u(\cdot)$ of (1.1) is "bad" in the sense that $\|u''\|_\infty$ is not bounded uniformly in ε, the standard h-version finite element method (FEM) generates huge errors through the whole domain when ε is very small. Typically, it is caused by a small interval of width $O(\varepsilon)$ or $O(\sqrt{\varepsilon})$ (called boundary layer), in which u'' rapidly changes.

To overcome this difficulty for the h-version finite element method, there are roughly two types of methods in the literature: 1) stabilize the approximation by modifying the variational form under quasi-uniform mesh (if we are only interested in the overall behavior of the solution); 2) use anisotropic meshes by putting more grid points in the boundary layer region (if we want to resolve the solution inside

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the boundary layer region as well). Many such schemes are extensively studied in the context of singularly perturbed problems since the 1970s; see [1], [3]–[20], and references therein. Among those, upwinding schemes and streamline diffusion finite element methods (SDFEM) are in the first group, while Bakhalov mesh [1] and Shishkin mesh [19] belong to the second group.

Given a partition (or grid)

$$\mathbb{T}^n = \{x_i |\, 0 = x_0 < x_1 < \cdots < x_{n+1} = 1\}, \tag{1.2}$$

we denote the FEM solution of (1.1) on $\mathbb{T}^n$ by u^n, and the interpolation of the exact solution on $\mathbb{T}^n$ by u_I^n. If there is only one boundary layer, both Bakhalov mesh and Shishkin mesh have $n + n$ grid points, with n uniform grids outside the boundary layer region and n grids inside boundary layer region. The n grids inside the boundary layer region are uniform for Shishkin mesh and properly graded for Bakhalov mesh. The *a priori* error estimates are

$$\|u - u^n\|_\infty \le Cn^{-2} \tag{1.3}$$

under Bakhalov mesh, and

$$\|u - u^n\|_\infty \le Cn^{-2} \ln^2 n \tag{1.4}$$

under Shishkin mesh. Here $C > 0$ is independent of ε, and the convergence rates are ε-uniform.

In this article, we propose and analyze a recovery method under uniform or quasi-uniform mesh. This recovery yields ε-uniform convergence in all elements, except the one containing the boundary layer. The convergence rate is the same as using Bakhalov mesh. In addition, we are able to locate a point in each element where the approximation is extremely accurate. The analysis in this article is elementary and the scheme is surprisingly simple. Here is our first algorithm.

Algorithm 1.

- Step 1. Solve the problem by the standard finite element method with n uniform grids. This step is likely to produce an oscillatory solution.
- Step 2. Add an extra grid point anywhere in the element containing the boundary layer, and solve the same problem again. This step produces another solution.
- Step 3. Find intersections of the two solutions in Step 1 and Step 2 in all elements and link those intersections by straight lines.

The above algorithm will produce a highly accurate solution in all but one element. The theoretical foundation will be discussed in Section 3. The key observation is that adding one grid point alters the direction of the oscillation. Therefore, the two solutions always have an intersection in each element except the one containing the boundary layer.

An astonishing discovery is that the intersection point in each element is invariant, i.e., it is independent of the location of the extra grid point in the boundary layer element (the element that contains an boundary layer) and independent of the number of grid points added to the boundary layer element. In other words, no matter how many grid points we add to the boundary layer element and no matter where we put them, those intersections outside the boundary layer element are always the same.

Now let us explain precisely the above description. Without loss of generality, we assume the boundary layer is at $x = 1$. Starting from $\mathbb{T}^n$, we add m points $s_1, \ldots, s_m$ arbitrarily in $(x_n, 1)$ and denote the new partition as $\mathbb{T}^{n+m}$. Then FEM