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CONVERGENCE AND SUPERCONVERGENCE OF A NONCONFORMING FINITE ELEMENT ON ANISOTROPIC MESHES

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Abstract. The main aim of this paper is to study the error estimates of a nonconforming finite element for general second order problems, in particular, the superconvergence properties under anisotropic meshes. Some extrapolation results on rectangular meshes are also discussed. Finally, numerical results are presented, which coincides with our theoretical analysis perfectly.

Key Words. nonconforming finite element, anisotropic meshes, superconvergence, extrapolation.

1. Introduction

It is well-known that regular assumption or quasi-uniform assumption [10, 13] of finite element meshes is a basic condition in the analysis of finite element approximation both for conventional conforming and nonconforming elements. However, with the advances of the finite element methods and its applications to other fields and more complex problems, the above regular or quasi-uniform assumption becomes quite a restriction in practice for some problems in the finite element methods. For example, the solution may have anisotropic behavior in part of the domain, that is to say, the solution varies significantly only in certain directions. Such problems are frequently encountered in perturbed convection-diffusion-reaction equations where boundary or interior layers appear. In such cases, it is more effective to use anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction. Consider a bounded convex domain $\Omega \subset R^2$. Let $\mathcal{J}_h$ be a family of meshes of Ω. Denote the diameter of an element K and the diameter of the inscribed circle of K by h_K and ρ_K, respectively. $h = \max\limits_{K\in\mathcal{J}_h} h_K$. It is assumed in the classical finite element theory that $\frac{h_K}{\rho_K} \leq C$, where C be a positive constant independent of K and the function considered. Such assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements are characterized by $\frac{h_K}{\rho_K} \to \infty$ as $h \to 0$. Some early papers have been written to prove error estimates under more general conditions (refer to [7, 25]). Recently, much attention is paid to FEMs with anisotropic meshes. In particular, for anisotropic rectangular meshes we refer to Acosta [1, 2], Apel [3, 4, 5, 6], Chen [16, 17, 31, 38], Duran [22, 23], Shenk [37] and references therein, and for narrow quadrilateral meshes to Zenisek [45]. But to our best knowledge, there are few papers focused on the nonconforming elements under anisotropic meshes.

On the other hand, researchers have observed that for certain classes of problems the rate of convergence of the finite element solution and/or its derivatives at some

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special points exceeds the best global rate. This phenomenon has been termed " superconvergence" and has been analyzed mathematically because of its practical importance in engineering computations. Also some postprocessing methods have been developed to improve the accuracy of finite element solution. Many superconvergence results about conforming FEMs have been obtained, see e.g., [14, 26, 29, 43]. Do the superconvergence results of conforming elements still hold for those nonconforming ones? The answer is affirmative. In [15, 39], the superconvergence of Wilson element is studied and the superconvergence estimate of the gradient error on the centers of elements is obtained. Recently, some superconvergentce results of rotated $\mathcal{Q}_1$ type elements are derived for quasi-uniform meshes in [30, 32]. On the other hand, Wang [44] proposed a least-square surface fitting method to obtain the superconvergence under quasi-uniform mesh assumption. The main feature of their method is to apply an L^2 projection on a coarser mesh with size $\tau = \mathrm{O}(h^\alpha)$ ($\alpha \in (0,1)$). At the same time, extrapolation is widely used in the finite element method. Interested readers are referred to [8, 9, 29, 35, 36] for extrapolation results of the conforming linear and bilinear element.

In this work, we first study the anisotropic interpolation error on anisotropic affine quadrilateral meshes of a five-node nonconforming finite element proposed by [24, 30], and the optimal consistency error is derived by a detailed analysis for anisotropic affine quadrilaterals, which extends the results of [38] for the rectangular meshes. We comment that since the interpolation of the original rotated $\mathcal{Q}_1$ element does not satisfy the anisotropic interpolation properties, reference [4] deals with the modified anisotropic rectangular rotated $\mathcal{Q}_1$ element with the shape space $span\{1, \xi, \eta, \xi^2\}$. There has been other works for the modified rotated $\mathcal{Q}_1$ element (cf. [5, 19, 28]).

In section §3, following the technique developed in [30, 38], we obtain a higher order $O(h^2)$ of consistency error under anisotropic rectangular meshes. Based on this fact and some other higher order error estimates proved in this section, a superconvergent approximation between the interpolation of the exact solution and the finite element solution is derived. Then a superconvergent estimate on the centers of elements is obtained, and the global superconvergence $O(h^2)$ for the gradient of the solution is also derived with the aid of a suitable postprocessing method.

In section §4, for regular rectangular meshes, we study some error expansions for the five-node nonconforming element. Based on these expansions, we obtain a sharp error estimates $O(h^3)$ by extrapolations. In the last section, some numerical examples are presented to validate our theoretical analysis.

Finally, we recall some notations and terminology (or refer to [10, 13]). Let $(\cdot,\cdot)$ denote the usual L^2 -inner product and $\|u\|_{r,p,\Omega}$ (resp. $|u|_{r,p,\Omega}$) be the usual norm (resp. semi-norm) for the Sobolev space $W^{r,p}(\Omega)$. When $p = 2$, denote $W^{2,r}(\Omega)$ by $H^r(\Omega)$. Throughout this paper, C will be used as a generic positive constant, which is independent of h_K, and may be independent of the aspect ratio $\frac{h_K}{\rho_K}$ in §2 and §3.

2. Error estimates for general second order problems under anisotropic affine quadrilaterals

Let $\widehat{K} = [-1,1] \times [-1,1]$ be the reference element. Its four vertices are: $\widehat{a}_1 = (-1,-1), \widehat{a}_2 = (1,-1), \widehat{a}_3 = (1,1), \widehat{a}_4 = (-1,1)$, and its four sides are $\widehat{l}_1 = \widehat{a}_1\widehat{a}_2, \widehat{l}_2 = \widehat{a}_2\widehat{a}_3, \widehat{l}_3 = \widehat{a}_3\widehat{a}_4, \widehat{l}_4 = \widehat{a}_4\widehat{a}_1$.

The nonconforming five-node element[24,30] $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ on $\widehat{K}$ is defined as follows:

$$\widehat{\Sigma} = \{\widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{v}_4, \widehat{v}_5\}, \qquad \widehat{P} = span\{1, \xi, \eta, \varphi(\xi), \varphi(\eta)\}, \tag{2.1}$$