INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 4, Number 1, Pages 39–47

LINEAR ADVECTION WITH ILL-POSED BOUNDARY CONDITIONS VIA L^1-MINIMIZATION

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Abstract. It is proven that in dimension one the piecewise linear best L^1-approximation to the linear transport equation equipped with a set of ill-posed boundary conditions converges in $W^{1,1}_{\text{loc}}$ to the viscosity solution of the equation and the boundary layer associated with the ill-posed boundary condition is always localized in one mesh cell, i.e., the "last" one.

Key Words. Finite elements, best L^1-approximation, viscosity solution, linear transport, ill-posed problem

1. Introduction

The goal of this paper is to explain a phenomenon that has been reported in [4]; namely, finite-element-based best L^1-approximations seem to converge to viscosity solutions of some classes of first-order PDE's. In particular we prove in this paper that it is indeed the case in dimension one for the linear transport equation equipped with a set of ill-posed boundary conditions.

To explain our interest for finite element best L^1-approximations and ill-posed boundary conditions, we now briefly recall the result from [4] that we exactly refer to. We denote by Ω a bounded domain of $\mathbb{R}^d$ with smooth boundary. Let $\alpha > 0$ be a real number and let $\boldsymbol{\beta} \in [\mathcal{C}^1(\overline{\Omega})]^d$ be a smooth vector field. Let u_0 be a smooth function on $\partial\Omega$, say $u_0 \in \mathcal{C}^2(\partial\Omega)$, and let $f \in W^{1,1}(\Omega)$. Following Bardos–le Roux–Nédélec, [2], we say that u is a viscosity solution to

$$\alpha u + \nabla\cdot(\boldsymbol{\beta} u) = f; \quad u|_{\partial\Omega} = u_0, \tag{1.1}$$

if $u \in \mathrm{BV}(\Omega)$, u solves the PDE, and u satisfies the boundary condition in the following sense

$$\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\mathbf{n})(u-k)(\mathrm{sg}(u-k) - \mathrm{sg}(u_0-k)) \geq 0, \quad \forall k \in \mathbb{R}, \tag{1.2}$$

where $\mathrm{sg}(t)$ is the sign of t if $t \neq 0$ and $\mathrm{sg}(0) = 0$. In the present linear case, the boundary condition amounts to enforcing $u = u_0$ on $\partial\Omega^- = \{\mathbf{x} \in \partial\Omega \,|\, \boldsymbol{\beta}(\mathbf{x})\cdot\mathbf{n}(x) < 0\}$.

Using arguments similar to those in [2] and [1], it is possible to prove that (1.1) has a unique viscosity solution provided α is large enough. The bulk of the argument consists of proving that the solution to the following problem

$$\alpha u_\epsilon + \nabla\cdot(\boldsymbol{\beta} u_\epsilon) - \epsilon\nabla^2 u_\epsilon = f; \quad u_\epsilon|_{\partial\Omega} = u_0, \tag{1.3}$$

Received by the editors December 21, 2005.
1991 *Mathematics Subject Classification.* 65N35, 65N22, 65F05, 35J05.
This material is based upon work supported by the National Science Foundation grant DMS-0510650.

converges in $L^1(\Omega)$ and the limit is the so-called viscosity solution, i.e. the limit satisfies the PDE (1.1) and the boundary condition (1.2).

Despite its appearance, the problem (1.1) is not purely formal. It arises when one tries to approximate (1.3) on finite element meshes that are not refined enough. For instance, denoting by h the mesh size, whenever $\epsilon/h^2 \ll \|\boldsymbol{\beta}\|_{L^\infty}/h$, the second-order term in (1.3) is completely dominated by the first-order one, and approximating (1.3) in this circumstance amounts to trying to solve (1.1), where the boundary condition is understood in the classical sense instead of (1.2).

It has been shown in [4] that the best L^2-approximation (i.e., Least-Squares) does not converge to the right limit of (1.3) under the limiting process $\lim_{h\to 0}\lim_{\epsilon\to 0}$. The situation is quite different in $L^1(\Omega)$, since for reasons that will be detailed latter, the best L^1-approximation to (1.1) converges to the viscosity solution. Before going into the details of the proof and to illustrate this claim, we now reproduce a numerical experiment reported in [4].

Consider the 2D rectangular domain $\Omega =]0,1[^2$ and set $\partial\Omega_D = \{x=0\}\cup\{x=1\}$ and $\partial\Omega_N = \{y=0\}\cup\{y=1\}$, i.e., We want to solve the following scalar problem

$$u + \partial_x u = 1; \quad u|_{\partial\Omega_D} = 0, \tag{1.4}$$

Of course the above problem is not well-posed in the usual sense, since the outflow boundary condition is over-specified, but it is meaningful in the viscosity sense. Let $\{X_h\}_{h>0}$ be a sequence of H^1-conforming finite element spaces constructed on a shape regular mesh family and such that for all v_h in X_h, $v_h|_{\partial\Omega_D} = 0$. We show in figure 1 the best L^1-approximation and the best L^2-approximation of the above problem using a coarse mesh, $h = 1/10$. The $\mathbb{P}_1$ Lagrange interpolant of the exact solution is shown in the left panel, the best L^1-approximation is in the center panel, and the best L^2-approximation is shown in the right panel. Considering the mesh

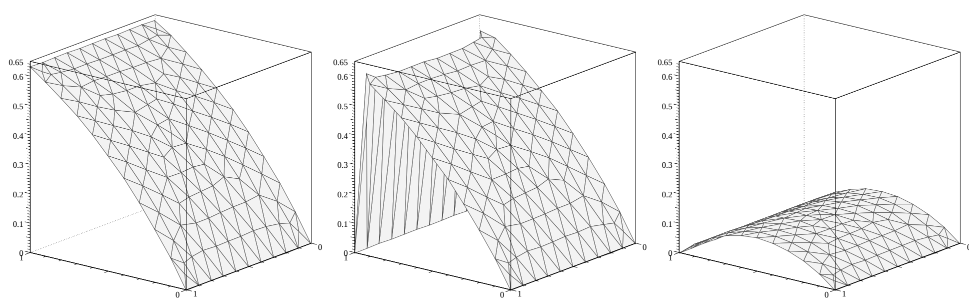

FIGURE 1. Viscosity solution to (1.4) from [4]. Left: $\mathbb{P}_1$ Lagrange interpolant of exact solution; center, L^1 solution; right, L^2 solution.

used, the best L^1-approximation is a reasonable approximation, whereas the Least-Squares solution is completely wrong. Contrary to what it looks, the two horn-like spikes observable on the graph of the L^1-solution are not over-shootings. These are perspective effects induced by the fact that the two corresponding $\mathbb{P}_1$-nodes are not aligned with the others. Given that the Least-Squares method together with its many variants is a central part for the stabilization of the Galerkin technique (see *e.g.* [3, 6, 7]), the above example gives new reasons why the Galerkin-Least-Squares method cannot generally cope properly with shocks and boundary layers without the help of shock-capturing terms [7, 5].

The rest of the paper is organized as follows. In §2 we introduce the ill-posed one-dimensional linear advection problem under scrutiny in this paper. The discrete