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MODELING, ANALYSIS AND DISCRETIZATION OF STOCHASTIC LOGISTIC EQUATIONS

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Abstract. The well–known logistic model has been extensively investigated in deterministic theory. There are numerous case studies where such type of nonlinearities occur in Ecology, Biology and Environmental Sciences. Due to the presence of environmental fluctuations and a lack of precision of measurements, one has to deal with effects of randomness on such models. As a more realistic modeling, we suggest nonlinear stochastic differential equations (SDEs)

$$dX(t) = [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]dt + \sigma X(t)^{\alpha}|K - X(t)|^{\beta} dW(t)$$

of Itô-type to model the growth of populations or innovations X, driven by a Wiener process W and positive real constants ρ, λ, K, μ, α, $\beta \geq 0$. We discuss well–posedness, regularity (boundedness) and uniqueness of their solutions. However, explicit expressions for analytical solution of such random logistic equations are rarely known. Therefore one has to resort to numerical solution of SDEs for studying various aspects like the time–evolution of growth patterns, exit frequencies, mean passage times and impact of fluctuating growth parameters. We present some basic aspects of adequate numerical analysis of these random extensions of these models such as numerical regularity and mean square convergence. The problem of keeping reasonable boundaries for analytic solutions under discretization plays an essential role for practically meaningful models, in particular the preservation of intervals with reflecting or absorbing barriers. A discretization of the continuous state space can be circumvented by appropriate methods. Balanced implicit methods (see Schurz, IJNAM 2 (2), p. 197-220, 2005) are used to construct strongly converging approximations with the desired monotone properties. Numerical studies can bring out salient features of the stochastic logistic models (e.g. almost sure monotonicity, almost sure uniform boundedness, delayed initial evolution or earlier points of inflection compared to deterministic model).

Key Words. logistic growth, stochastic logistic equation, properties of solutions, numerical methods, balanced implicit methods, boundedness, convergence, stability, monotonicity

1. Introduction

Logistic growth phenomenon is observed in numerous models and underlying data such as for the population of fruit flies or flour beetle in population ecology or innovation diffusion in marketing sciences or social sciences. In the continous time

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framework it is commonly believed that the evolution of the number $x = x(t)$ of certain species can be approximately modeled by the per-capita-growth rate

$$\frac{1}{x}\frac{dx}{dt} = \lambda(K - x) - \mu \tag{1}$$

where $\lambda > 0$ is some growth parameter, μ the death rate and $K > 0$ the underlying carrying capacity limited by a finite number of natural resources. Model (1) with $\mu = 0$ is also known as *Verhulst-Pearl equation*, see [34], [46]. It was used by several authors to describe the evolution of species, populations or innovations, see [12], [35], [8], [22] or [37], and specified later by [20], [21], [26] and [47] for biological applications with delay effects, among many others.

It is well-known that equation (1) has two equilibria solutions, namely a locally asymptotically unstable solution $x_1^* = 0$ and, if $\mu = 0$, a globally asymptotically stable $x_2^* = K$. Moreover, these points represent barriers for any other solution and, if $\mu = 0$, the interval $(0, K)$ is left invariant and attracting from above by the related flow of analytical solutions. Furthermore, discrete analoga are often used to motivate the existence and effects of chaos in related dynamical systems.

In reality of collecting and analyzing environmental data, these models need to be specified. In particular, due to the Heisenberg's uncertainty principle and the resulting lack of precise measurements, the logistic growth undergoes environmental and parametric noise. Recall that Heisenberg's uncertainty principle also means that two or more quantities (here our model parameters) cannot be estimated exactly, only with random deviations. Consequently, meaningful stochastic generalizations of logistic equations lead to nonlinear **s**tochastic **d**ifferential **e**quations (**SDEs**) of Itô-type

$$dX(t) = [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\,dt + \sigma X(t)^{\alpha}|K - X(t)|^{\beta} dW(t) \tag{2}$$

driven by a standard Wiener process $(W(t) : t \geq 0)$, started at $X_0 \in \mathbb{D} = [0, K] \subset \mathbb{R}^1$, where $\rho, \lambda, K, \mu, \sigma$ are positive and α, β nonnegative real parameters. There ρ can be understood as coefficient of transition (self-innovation), λ as coefficient of immitation depending on the contact itensity with its environment, K as a somewhat "optimal" environmental carrying capacity and μ as natural death rate. However, in view of issues of practical meaningfulness, model (2) makes only sense within deterministic algebraic constraints, either given by extra boundary conditions or self-inherent properties resulting into natural barriers at 0 at least. This fact is supported by the limited availability of natural resources as known from the evolution of species in population ecology. In what follows we study **almost sure regularity** (boundedness on $\mathbb{D}$) of both exact and numerical solutions of (2) which has been mostly omitted in literature in the latter case. At the same time we are aiming at the **maintenance of** certain **convergence** orders of related standard numerical approximations towards exact solutions. In particular we shall construct a numerical solution which exclusively possesses values in $\mathbb{D}$ and is mean square converging with order $\gamma = 0.5$ towards the exact solution. Note that usual numerical methods as most–used Euler method fail to live a.s. on bounded domain $\mathbb{D}$ for any choice of constant step sizes (for examples, see [38], [39]). Besides, "higher order methods" as systematically developed by [48] can not be applied in general, since their mathematical justification requires too much boundedness and smoothness on drift and diffusion coefficients of SDEs, which is not given within the general framework of model (2). The latter statement does not mean that we do not advise to try out methods of higher order of convergence in specific situations. It is more the expression for a current lack of knowledge on qualitative behavior of