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STABILITY-PRESERVING FINITE-DIFFERENCE METHODS FOR GENERAL MULTI-DIMENSIONAL AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. General multi-dimensional autonomous dynamical systems and their numerical discretizations are considered. Nonstandard stability-preserving finite-difference schemes based on the θ-methods and the second-order Runge-Kutta methods are designed and analyzed. Their elementary stability is established theoretically and is also supported by a set of numerical examples.

Key Words. finite-difference, nonstandard scheme, elementary stability, dynamical systems.

1. Introduction

The increasing study of realistic mathematical models in biology, ecology and medicine is a reflection of their use in helping to understand the dynamic processes involved in such areas as predator-prey and competition interactions, infectious diseases control and multi-species marine societies. Mathematical models usually consist of systems of differential equations that represent the rates of change of the size of each interacting component. In most of the interactions modeled all rates of change are assumed to be time independent, which makes the corresponding systems autonomous.

Numerical methods that approximate continuous dynamical systems are expected to be consistent with the original differential system, to be zero-stable and convergent. Nonstandard finite difference techniques, developed by Mickens [12, 14], have laid the foundation for designing methods that preserve the physical properties, especially the stability properties of equilibria, of the approximated differential system. Anguelov and Lubuma [1] have used Mickens' techniques to design nonstandard versions of the explicit and implicit Euler and the second order Runge-Kutta methods, under the limiting condition that all eigenvalues of the Jacobian at each equilibrium of the original differential system (for simplicity, we name those eigenvalues "equilibria"-eigenvalues) are single and real. However, a wide range of mathematical models do not satisfy the aforementioned limitation. Among them are most of the non-conservative predator-prey systems such as the Lotka-Volterra models [9, 17, 13], most models with Michaelis-Menten functional responses [11], the ratio-dependent models [8, 5], some SI, SIS and SIR epidemiology models [7, 15, 4] and most phytoplankton-nutrient systems [16, 6]. Therefore developing stability-preserving numerical methods for general autonomous dynamical systems

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that have not only single and real but also multiple real and complex "equilibria"-eigenvalues is of critical importance. Dimitrov and Kojouharov [3] have designed a variety of such nonstandard finite-difference schemes for general two-dimensional systems, based on the explicit Euler, the implicit Euler and the second-order Runge-Kutta methods. Lubuma and Roux [10] have constructed nonstandard numerical schemes, based on the θ-methods, that preserve the stability of equilibria for multi-dimensional systems having all of their "equilibria"-eigenvalues in a subregion of the complex plane. In this paper we extend the above nonstandard θ-methods and also develop a new class of stability-preserving nonstandard finite-difference schemes, based on the second-order Runge-Kutta methods, for multi-dimensional autonomous dynamical systems with arbitrary complex "equilibria"-eigenvalues. The proposed new elementary stable nonstandard (ESN) numerical schemes work very well with conservative as well as with non-conservative dynamical systems.

The paper is organized as follows. In Section 2 we provide some definitions and preliminary results. We state our main results in Section 3 and prove them in Section 4. In the last two sections we illustrate our theoretical results by numerical examples and outline some future research directions.

2. Definitions and Preliminaries

A general n-dimensional autonomous system has the following form:

$$\frac{dy}{dt} = f(y); \quad y(t_0) = y_0, \tag{1}$$

where $y = [y^1, y^2, \ldots, y^n]^T : [t_0, T) \to \mathbb{R}^n$, the function $f = [f^1, f^2, \ldots, f^n]^T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is differentiable and $y_0 \in \mathbb{R}^n$. The equilibrium points of System (1) are defined as the solutions of $f(y) = 0$.

Definition 1. *Let y^* be an equilibrium of System (1), $J(y^*)$ be the Jacobian of System (1) at y^* and $\sigma(J(y^*))$ denotes the spectrum of $J(y^*)$. An equilibrium y^* of System (1) is called linearly stable if $Re(\lambda) < 0$ for all $\lambda \in \sigma(J(y^*))$ and linearly unstable if $Re(\lambda) > 0$ for at least one $\lambda \in \sigma(J(y^*))$.*

A one-step numerical scheme with a step size h, that approximates the solution $y(t_k)$ of System (1) can be written in the form:

$$D_h(y_k) = F_h(f; y_k), \tag{2}$$

where $D_h(y_k) \approx \dfrac{dy}{dt}$, $F_h(f; y_k) \approx f(y)$ and $t_k = t_0 + kh$.

Definition 2. *Let y^* be a fixed point of the scheme (2) and the equation of the perturbed solution $y_k = y^* + \epsilon_k$ be linearly approximated by*

$$D_h \epsilon_k = J_h \epsilon_k, \tag{3}$$

where the right-hand side is the linear term in ϵ_k of the Taylor expansion of $F_h(f; y^* + \epsilon_k)$ around y^*. The fixed point y^* is called stable if $\|\epsilon_k\| \to 0$ as $k \to \infty$, and unstable otherwise, where ϵ_k is the solution of Equation (3).

Definition 3. *The finite-difference method (2) is called elementary stable if, for any value of the step size h, the linear stability of each equilibrium y^* of System (1) is the same as the stability of y^* as a fixed point of the discrete method (2).*

We introduce the nonstandard one-step finite-difference method based on a definition given by Anguelov and Lubuma [1].