ANALYSIS OF THE \([L^2, L^2, L^2]\) LEAST-SQUARES FINITE ELEMENT METHOD FOR INCOMPRESSIBLE OSEEN-TYPE PROBLEMS

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Dedicated to Professor Max D. Gunzburger on the occasion of his 60th birthday

Abstract. In this paper we analyze several first-order systems of Oseen-type equations that are obtained from the time-dependent incompressible Navier-Stokes equations after introducing the additional vorticity and possibly total pressure variables, time-discretizing the time derivative and linearizing the non-linear terms. We apply the \([L^2, L^2, L^2]\) least-squares finite element scheme to approximate the solutions of these Oseen-type equations assuming homogeneous velocity boundary conditions. All of the associated least-squares energy functionals are defined to be the sum of squared \(L^2\) norms of the residual equations over an appropriate product space. We first prove that the homogeneous least-squares functionals are coercive in the \(H^1 \times L^2 \times L^2\) norm for the velocity, vorticity, and pressure, but only continuous in the \(H^1 \times H^1 \times H^1\) norm for these variables. Although equivalence between the homogeneous least-squares functionals and one of the above two product norms is not achieved, by using these a priori estimates and additional finite element analysis we are nevertheless able to prove that the least-squares method produces an optimal rate of convergence in the \(H^1\) norm for velocity and suboptimal rate of convergence in the \(L^2\) norm for vorticity and pressure. Numerical experiments with various Reynolds numbers that support the theoretical error estimates are presented. In addition, numerical solutions to the time-dependent incompressible Navier-Stokes problem are given to demonstrate the accuracy of the semi-discrete \([L^2, L^2, L^2]\) least-squares finite element approach.

Key Words. Navier-Stokes equations, Oseen-type equations, finite element methods, least squares.

1. Problem formulation

As a first step towards the finite element solution of the time-dependent incompressible Navier-Stokes problem by using the least-squares principles, in this paper we analyze the \([L^2, L^2, L^2]\) least-squares finite element approximations to several first-order systems of Oseen-type equations all equipped with the homogeneous velocity boundary conditions. These systems are obtained from the time-dependent incompressible Navier-Stokes problem after introducing the additional vorticity and possibly total pressure variables, time-discretizing the time derivative and linearizing the non-linear terms.
We start with the derivation of these first-order Oseen-type problems and introduce some background and notations. Let $\Omega$ be an open bounded and connected domain in $\mathbb{R}^N$ ($N = 2$ or $3$) with Lipschitz boundary $\partial\Omega$. The time-dependent incompressible Navier-Stokes problem on the bounded domain $\Omega$ can be posed as the following initial-boundary value problem (cf. [13, 14, 15]):

Find $u(x, t) : \overline{\Omega} \times (0, T) \to \mathbb{R}^N$ and $p(x, t) : \overline{\Omega} \times [0, T) \to \mathbb{R}$ such that

$$
\begin{align*}
\frac{\partial u}{\partial t} - \frac{1}{\lambda} \Delta u + (u \cdot \nabla) u + \nabla p &= f & \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0 & \text{in } \Omega \times (0, T), \\
u &= 0 & \text{on } \partial\Omega \times [0, T], \\
u(\cdot, 0) &= u_0(\cdot) & \text{in } \Omega,
\end{align*}
$$

(1.1)

where the symbols $\Delta$, $\nabla$ and $\nabla \cdot$ stand for the Laplacian, gradient and divergence operators with respect to the spatial variable $x$, respectively; $u = (u_1, \ldots, u_N)^T$ is the velocity vector; $p$ is the pressure; $\lambda \geq 1$ is the Reynolds number and may be identified with the inverse viscosity constant $1/\nu$; $[0, T]$ is the time interval under consideration; $f = (f_1, \ldots, f_N)^T : \Omega \times (0, T) \to \mathbb{R}^N$ is a given vector function representing the density of body force; the initial velocity $u_0 : \overline{\Omega} \to \mathbb{R}^N$ with $u_0 = 0$ on $\partial\Omega$ is prescribed. All of them are assumed to be non-dimensionalized.

We now introduce some notations that are used throughout the article. When $N = 2$, we define the curl operator, $\nabla \times$, with respect to the spatial variable $x$ for a smooth scalar function $v$ by

$$
\nabla \times v = \left( \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \right)^T,
$$

and for a smooth 2-component vector function $v = (v_1, v_2)^T$ by

$$
\nabla \times v = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.
$$

When $N = 3$, we define the curl of a smooth 3-component vector function $v = (v_1, v_2, v_3)^T$ by

$$
\nabla \times v = \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)^T.
$$

We also define the following cross products. If $w$ is a scalar function and $v = (v_1, v_2)^T$, then

$$
w \times v = -v \times w = (-w v_2, w v_1)^T.
$$

If $w = (w_1, w_2, w_3)^T$ and $v = (v_1, v_2, v_3)^T$, then

$$
w \times v = (w_2 v_3 - w_3 v_2, w_3 v_1 - w_1 v_3, w_1 v_2 - w_2 v_1)^T.
$$

With these notations, it can be easily checked that the following identities hold: for a smooth vector function $u = (u_1, \ldots, u_N)^T$,

$$
\nabla \times (\nabla \times u) = -\Delta u + \nabla (\nabla \cdot u)
$$

(1.2)

and

$$
(w \times v) \cdot v = 0
$$

(1.3)

for $w = (w_1, \ldots, w_{2N-3})^T$ and $v = (v_1, \ldots, v_N)^T$.

Introducing the additional vorticity variable $\omega$ (cf. [2, 7, 10]),

$$
\omega = \nabla \times u \quad \text{on } \overline{\Omega} \times [0, T],
$$