FINITE ELEMENT APPROXIMATION OF THE NON-ISOTHERMAL STOKES-OLDROYD EQUATIONS

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Dedicated to Max Gunzburger on the occasion of his 60th birthday.

Abstract. We consider the Stokes-Oldroyd equations, defined here as the Stokes equations with the Newtonian constitutive equation explicitly included. Thus a polymer-like stress tensor is included so that the dependent variable structure of a viscoelastic model is in place. The energy equation is coupled with the mass, momentum, and constitutive equations through the use of temperature-dependent viscosity terms in both the constitutive model and the momentum equation. Earlier works assumed temperature-dependent constitutive (polymer) and Newtonian (solvent) viscosities when describing the model equations, but made the simplifying assumption of a constant solvent viscosity when carrying out analysis and computations; we assume no such simplification. Our analysis coupled with numerical solution of the problem with both temperature-dependent viscosities distinguishes this work from earlier efforts.

Key Words. viscous fluid, non-isothermal, finite elements, Stokes-Oldroyd.

1. Introduction

Viscoelastic flows occur in a variety of applications, including polymer processing. The complexity of the governing equations and the physical domains makes analysis of the mathematical models and the associated numerical methods especially difficult. Current efforts to model viscoelastic flows often revolve around the solution of a (modified) Stokes problem, [5]. The isothermal linear elasticity equations, modified in form to have the same dependent variable structure as the equations governing viscoelastic flows, is analyzed along with a numerical solution in [2]. The Stokes problem is a special case (the incompressible limit) of the equations considered in that work.

The purpose of this paper is to analyze the finite element solution of the non-isothermal Stokes problem, modified similarly as in [2]. Thermodynamics play a prominent role in many viscoelastic flow scenarios, especially in polymer processing. Realistic models must ultimately include temperature dependence, since flow characteristics such as viscosity vary widely as temperature varies within normal operating constraints, [1].

The rest of this paper is outlined as follows. The governing equations are presented in the next section, with particular attention given to the manner in which temperature dependence is expressed. Details regarding the weak formulation and

corresponding function spaces are provided in Section 3. In Section 4, the finite element formulation is developed along with an existence result for the finite element solution. Convergence results for the finite element solution are derived in Section 5, and numerical confirmation of these results are presented in Section 6. The paper concludes with a summary and a discussion of continuing work.

2. Governing Equations

We consider fluid flowing through a bounded, connected domain $\Omega \subset R^{\acute{d}}$, whose boundary we denote as Γ. Let the velocity be denoted by $\mathbf{u}$, pressure p, extra stress $\underline{\sigma}$, temperature T, and unit outward normal to the boundary $\mathbf{n}$. For viscoelastic fluid flow, the extra stress tensor is often split into a solvent and polymer part,

$$\underline{\sigma} = \underline{\sigma}_s + \underline{\sigma}_p.$$

Normally the solvent part of the extra stress is assumed to be Newtonian, i.e.

$$\underline{\sigma}_s = 2\frac{\eta_s(T)}{\eta_0(T_R)}d(\mathbf{u}),$$

where the rate-of-deformation tensor $d(\mathbf{u})$ is defined as

$$d(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right),$$

η_S is the solvent viscosity which depends at most on the temperature, and $\eta_0(T_R)$ is the zero-shear viscosity at a reference temperature T_R. A nonlinear differential or integral constitutive model is imposed for the polymer part $\underline{\sigma}_p$, [1]. As in [2], we simplify the constitutive model to a Newtonian relationship, and include this equation explicitly to preserve the dependent variable structure associated with viscoelastic constitutive models, such as Giesekus or Oldroyd-B. Whereas only the isothermal case is considered in [2], we analyze the case where both $\underline{\sigma}_p$ and $\underline{\sigma}_s$ depend on temperature. Specifically, we assume that

(2.1) $\underline{\sigma}_p - 2\alpha_1(T)d(\mathbf{u}) = \underline{0},$

where an Arrhenius equation characterizes the dependence of polymer viscosity (also scaled to $\eta_0(T_R)$) upon temperature, i.e.

$$\alpha_1(T) = A_1 \exp\left(\frac{B_1}{T}\right),$$

and $B_1 \neq 0$. The coefficients A_1 and B_1 are defined so that $0 < \alpha_1(T) \le 1$. We assume the existence of maximum and minimum values for the viscosity

(2.2) $\alpha_{1,min} \le \alpha_1(T) \le \alpha_{1,max}.$

The (scaled) solvent viscosity is defined in a similar manner so that

(2.3) $\underline{\sigma}_s - 2\epsilon\alpha_2(T)d(\mathbf{u}) = \underline{0},$

with

$$\alpha_2(T) = A_2 \exp\left(\frac{B_2}{T}\right).$$

Once again we choose A_2 and B_2 so that $0 < \alpha_2(T) \le 1$ and

(2.4) $\alpha_{2,min} \le \alpha_2(T) \le \alpha_{2,max},$

but here we may have $B_2 = 0$. The definition of $\underline{\sigma}_s$ includes ϵ because the solvent part of the viscosity is assumed to be much smaller than the polymer part. Furthermore, the term $2\epsilon\alpha_2(T)d(\mathbf{u})$ has special significance in that it increases stability. Hence the parameter ϵ is considered a penalty parameter, and is assumed small.