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REMARKS ON CONTROLLABILITY OF THE ANYSOTROPIC LAMÉ SYSTEM

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Abstract. In this paper we established a Carleman estimate for the elasticity system with the residual stresses. As an application of this estimate we obtain exact controllability results for the same system with locally distributed control.

Key Words. Controllability, anysotropic Lamé system

1. Introduction

Let us denote $x = (x_0, x')$, where x_0 (resp. x') stands for the time (resp. spatial) variable. This paper is concerned with global Carleman estimates for the Lamé system

$$\rho \partial_{x_0}^2 u_i - \sum_{j=1}^{3} \partial_{x_i}(\sigma_{ij}) = f_i \quad \text{in } Q,\ 1 \le i \le 3, \tag{1}$$

where Ω is a bounded domain with boundary $\partial\Omega \in C^3$, $Q = (0,T)\times\Omega$, $\mathbf{u}(x) = (u_1, u_2, u_3)$ is the displacement, $\mathbf{f} = (f_1, f_2, f_3)$ is the density of external forces and σ_{ij} is the stress tensor:

$$\sigma_{ij} = a_{ijhk}(x)\partial_{x_h} u_k.$$

On the boundary, we equip the Lamé system with zero Dirichlet boundary conditions:

$$\mathbf{u} = 0 \quad \text{on } \Sigma,$$

where we have denoted $\Sigma = (0,T)\times\partial\Omega$.

We introduce the following standard assumptions on the coefficients a_{ijhk}

$$\begin{cases} a_{ijhk} = a_{jikh} = a_{hkij}, \\ a_{ijhk}X_{ij}X_{kh} \ge \alpha X_{ij}X_{ij} \quad \forall X \in \mathbb{R}^9 \ \text{ with } \ X_{ij} = X_{ji}, \end{cases} \tag{2}$$

where α is some positive number.

In this paper we will strict to the case of the anisotropic Lamé system with residual free stresses:

$$\begin{aligned} \sigma = \mathbf{R} + (\nabla\mathbf{u})\mathbf{R} + \lambda(tr\epsilon)E_3 + 2\mu\epsilon + \beta_1(tr\epsilon)(tr\mathbf{R})E_3 + \beta_2(tr\mathbf{R})\epsilon \\ + \beta_3((tr\epsilon)\mathbf{R} + tr(\epsilon\mathbf{R})E_3) + \beta_4(\epsilon\mathbf{R} + \mathbf{R}\epsilon), \end{aligned} \tag{3}$$

where E_3 ia a unit matrix,

$$\epsilon = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$$

and

$$\nabla\cdot\mathbf{R} = 0 \quad \text{in } Q. \tag{4}$$

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We will assume for simplicity that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ and therefore

$$\sigma = \mathbf{R} + (\nabla \mathbf{u})\mathbf{R} + \lambda(tr\epsilon)E_3 + 2\mu\epsilon, \tag{5}$$

for some λ, μ and $\mathbf{R}$. We will impose the following regularity assumptions on the Lamé coefficients

(6)

$$\rho,\ \lambda,\ \mu,\ R_{ij} \in C^2(\overline{\Omega})\ i,j \in \{1,2,3\}, \quad \rho > 0,\ \ \mu E_3 - \mathbf{R}\ \text{ and }\ (\lambda + 2\mu)E_3 - \mathbf{R} > 0$$

positive definite in Ω.

The first goal of this paper is to establish appropriate global Carleman estimates for the Lamé system with residual free stress. For displacements $\mathbf{u}$ with compact support, such estimates were obtained in the previous works [27], [25], [18]. More results are available for the isotropic Lamé system. Thus, in the stationary case we refer to Dehman-Robbiano [8] and Weck [31] for displacements with compact support and Imanuvilov-Yamamoto [17] for displacements satisfying Dirichlet boundary conditions. For the nonstationary isotropic Lamé system, see [10] for displacements with compact support and [14]–[16] in the other case. In this paper we have extended the techniques in [15] to consider the anisotropic Lamé system (1) with σ given by (5). Our Carleman estimates will hold for displacements $\mathbf{u}$ satisfying zero Dirichlet coditions on Σ.

The last section of this paper is devoted to the exact controllability of the Lamé system. To our best knowledge, the first observability result for the Lamé system was proved in [24] using multipliers of the form $(x_i - x_i^0)\frac{\partial u}{\partial x_i}$, which led to the observability inequality

$$E(x_0) = \int_\Omega (\mu|\nabla \mathbf{u}|^2 + (\lambda+\mu)|\nabla \cdot \mathbf{u}|^2)dx' \le C \int_{(0,T)\times\Gamma_0} \left|\frac{\partial \mathbf{u}}{\partial n}\right|^2 d\Sigma$$

when the Lamé coefficients μ and λ are constants. Here, $E(t)$ is the energy. The control is exerted at $(0,T)\times\Gamma_0$ where $\Gamma_0 \subset \partial\Omega$ and homogeneous Dirichlet boundary conditions are assumed for u. Further results have been deduced in [1] by Alabau and Komornik for the anisotropic Lamé system by essentially the same multipliers method. Several questions concerning the approximate controllability/uniqueness of the Lamé system were studied in [10] by Eller, Isakov, Nakamura and Tataru by means of Carleman estimates. They obtained approximate controllability with a control distributed over any open subset of the boundary for a sufficiently large time. A series of important results have been obtained quite recently in works of Bellassoued [5]–[7]. In particular, he has proved a "logarithmic type" energy decay estimate in the case where the geometrical control condition of Bardos, Lebeau, Rauch is not fulfilled. An interesting result was also proved by Zuazua in [32], for the isotropic Lamé system with the locally distributed control $\mathbf{v}$ of the form $\chi_\omega \mathbf{v}$, where $\mathbf{v} = (v_1, \ldots, v_n)$ and $v_n \equiv 0$. Under some geometric assumptions on the domain D he established the approximate controllability for the isotropic Lamé system.

Several works are devoted to the construction of dissipative "feedback" boundary conditions for the Lamé system. In [2], Alabau and Komornik introduced dissipative boundary conditions of the form $\sigma(\mathbf{u})n + A\mathbf{u} + B\partial_{x_0}\mathbf{u} = 0$ on the controlled part of the boundary Γ_0. Under some geometric conditions on Γ_0, they established the exponential decay of the energy

$$E(x_0) = \frac{1}{2}\int_\Omega (|\partial_{x_0}\mathbf{u}|^2 + \sigma_{ij}\epsilon_{ij}(\mathbf{u}))dx' + \frac{1}{2}\int_{\Gamma_0} A|\mathbf{u}|^2 dS \le e^{-\omega x_0}$$