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CONFORMING CENTROIDAL VORONOI DELAUNAY TRIANGULATION FOR QUALITY MESH GENERATION

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This paper is dedicated to Max Gunzburger on the occasion of his 60th birthday.

Abstract. Although the methodology of centroidal Voronoi tessellation (CVT) has been widely used for mesh generation on complex geometries, a clear characterization of the influence of geometric constraints on the CVT-based meshing is still lacking. In this paper, we first give a clear definition of the conforming centroidal Voronoi Delaunay triangulation (CCVDT) and then propose an efficient algorithm for its construction in two dimensional space. Finally, we show the high-quality of CCVDT meshes and the effectiveness and robustness of our algorithm through extensive examples.

Key Words. centroidal Voronoi tessellation, Delaunay triangulation, mesh generation.

1. Introduction

Mesh generation often forms a crucial part of the numerical solution procedure in many applications. In the past few decades, automatic, unstructured mesh generations for complex 2D/3D domains have provided very successful tools for solving complex application problems, some of those well-studied techniques include AFT [22, 24], Octree [26], Voronoi/Delaunay-based methods [1, 3–5, 27, 30], and DistMesh [25]. It is well known that the quality of Delaunay-based triangular/tetrahedral meshes is greatly affected by the placement of the generating points of the associated Voronoi regions. Many work has been devoted to finding robust and efficient algorithms to distribute the generating points by some optimal criteria, for example, the Laplacian smoothing [16], the centroidal Voronoi tessellation (CVT) [9] and the optimal Delaunay triangulation (ODT) [7]. We here are specially interested in the first approach.

Centroidal Voronoi tessellation proposed in [9] is a special Voronoi tessellation having the property that the generators of the Voronoi diagram are also the centers of mass, with respect to a given density function, of the corresponding Voronoi cells. CVTs are very useful in many applications, including but not limited to image and data analysis, vector quantization, resource optimization, design of experiments, optimal placement of sensors and actuators, cell biology, territorial behavior of animals, numerical partial differential equations, point sampling, meshless computing, mesh generation and optimization, reduced-order modeling, computer graphics, and mobile sensing networks. Recently, CVT and its duality centroidal Voronoi Delaunay triangulation (CVDT) based mesh generation and mesh optimization have attracted a lot of attention and been used for many applications due to its optimal

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properties, see [1, 2, 10, 11, 13–15]. For example, in the medical imaging and simulation for surgical operations whcich are used to not just diagnose the patient's ailments but also test alternative treatments, a mesh generator with good triangle/tetrahedron quality and robust control over the mesh sizing and the number of elements is desired for numerical simulations [29].

In this paper, we propose an effective and efficient algorithm for high-quality triangular mesh generation based on the CVT-methodology, specifically, the conforming centroidal Voronoi Delaunay triangulation (CCVDT). The plan of the rest of the paper is as follows. We first give a brief introduction to the concept of CVT in Section 2.1, and then generalize the definition for conforming mesh generation of complicated geometries in Section 2.2. We also propose an algorithm for approximate CCVDT mesh construction in two dimensional space in Section 3, and some mesh examples generated by our algorithm for different geometries are given in Section 4 to show the high quaity of CCVDT meshes. Finally we make some concluding remarks in Section 5.

2. Conforming Centroidal Voronoi Delaunay Triangulation

2.1. Centroidal Voronoi tessellation. Given an open bounded domain $\Omega \in \mathbb{R}^d$ and a set of distinct points $\{\mathbf{x}_i\}_{i=1}^n \subset \Omega$. For each point $\mathbf{x}_i$, $i = 1, \ldots, n$, define the corresponding Voronoi region V_i, $i = 1, \ldots, n$, by

$$V_i = \big\{\mathbf{x} \in \Omega \quad | \quad \|\mathbf{x} - \mathbf{x}_i\| < \|\mathbf{x} - \mathbf{x}_j\| \quad \text{for } j = 1, \cdots, n \quad \text{and} \quad j \neq i\big\} \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean distance in $\mathbb{R}^d$. Clearly $V_i \cap V_j = \emptyset$ for $i \neq j$, and $\cup_{i=1}^n \overline{V}_i = \Omega$ so that $\{V_i\}_{i=1}^n$ is a tessellation of Ω. We refer to $\{V_i\}_{i=1}^n$ as the *Voronoi tessellation* (VT) of Ω associated with the point set $\{\mathbf{x}_i\}_{i=1}^n$. A point $\mathbf{x}_i$ is called a *generator*; a subdomain $V_i \subset \Omega$ is referred to as the *Voronoi region* corresponding to the generator $\mathbf{x}_i$. It is well-known that the dual tessellation (in a graph-theoretical sense) to a Voronoi tessellation of Ω is the so-called *Delaunay triangulation* (DT). The Voronoi regions V_i's are convex polygons if Ω is convex and their vertices consist of circumcenters of the corresponding Delaunay triangles.

Given a density function $\rho(\mathbf{x}) \geq 0$ defined on Ω, for any region $V \subset \Omega$, define $\mathbf{x}^*$, the *mass center* or *centroid* of V by

$$\mathbf{x}^* = \frac{\displaystyle\int_V \mathbf{y}\rho(\mathbf{y})\, \mathrm{d}\mathbf{y}}{\displaystyle\int_V \rho(\mathbf{y})\, \mathrm{d}\mathbf{y}}. \tag{2}$$

Then a special family of Voronoi tessellations is defined in the following [9]:

Definition 1. *We refer to a Voronoi tessellation* $\{(\mathbf{x}_i, V_i)\}_{i=1}^n$ *of* Ω *as a* centroidal Voronoi tessellation *if and only if the points* $\{\mathbf{x}_i\}_{i=1}^n$ *which serve as the generators of the associated Voronoi regions* $\{V_i\}_{i=1}^n$ *are also the centroids of those regions, i.e., if and only if we have that*

$$\mathbf{x}_i = \mathbf{x}_i^* \quad \textit{for} \ \ i = 1, \ldots, n\,. \tag{3}$$

The corresponding Delaunay triangulation is then called a centroidal Voronoi Delaunay triangulation.

General Voronoi tessellations do not satisfy the CVT property. It is worth noting that CVT or CVDT may not be unique [9]. The CVT concept also can be generalized to very broad settings that range from abstract spaces and distance metrics to discrete point sets [9, 11].