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NUMERICAL ANALYSIS OF A HIGHER ORDER TIME RELAXATION MODEL OF FLUIDS

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We dedicate this paper to Max Gunzburger on the occasion of his 60th birthday

Abstract. We study the numerical errors in finite element discretizations of a time relaxation model of fluid motion:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} + \chi \mathbf{u}^* = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0$$

In this model, introduced by Stolz, Adams and Kleiser, $\mathbf{u}^*$ is a generalized fluctuation and χ the time relaxation parameter. The goal of inclusion of the $\chi \mathbf{u}^*$ is to drive unresolved fluctuations to zero exponentially. We study convergence of discretization of the model to the model's solution as h, $\Delta t \to 0$. Next we complement this with an experimental study of the effect the time relaxation term (and a nonlinear extension of it) has on the large scales of a flow near a transitional point. We close by showing that the time relaxation term does not alter shock speeds in the inviscid, compressible case, giving analytical confirmation of a result of Stolz, Adams and Kleiser.

Key Words. time relaxation, deconvolution, turbulence

1. Introduction

A fluid's velocity at higher Reynolds numbers contains many spatial scales not economically resolvable on computationally feasible meshes. For this reason, many turbulence models, large eddy simulation models, numerical regularization and computational stabilizations have been explored in computational fluid dynamics. One of the simplest such regularization and most recent has been proposed by Adams, Stoltz and Kleiser [1, 2]. Briefly, if $\mathbf{u}$ represents the fluid velocity, h the characteristic mesh width, and $\delta = O(h)$ a chosen length scale, let $\mathbf{u}^*$ denote some representation of the part of $\mathbf{u}$ varying over length scales $< O(\delta)$, i.e. the fluctuating part of $\mathbf{u}$. (This will be made specific in Section 2.) The fluid regularization model of Adams, Stoltz and Kleiser, considered herein, arises by adding a simple, linear, lower order time regularization term, $\chi \mathbf{u}^*$, (where $\chi > 0$ has units of $1/time$) to the Navier-Stokes equations, giving:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} + \chi \mathbf{u}^* = \mathbf{f}\,, \in \Omega\,, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0\,, \in \Omega\,. \tag{1.2}$$

The term $\chi \mathbf{u}^*$ is intended to drive unresolved velocity scales to zero exponentially fast. Adams, Kleiser and Stoltz have performed extensive computational tests of this time relaxation model on compressible flows with shocks and on turbulent flows, for example, [1, 2] as has Guenanff [7] on aerodynamic noise. The originating study of (1.1),(1.2) was the work of Rosenau [11] and Schochet and Tadmor [12]

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in which the time relaxation model was developed from a regularized Chapman-Enskog expansion of conservation laws. Most recently, in [10] it was shown that at high Reynolds number, solutions to (1.1),(1.2), possess an energy cascade which terminates at the mesh scale δ with the proper choice of relaxation coefficient χ.

Our goal in this report is to connect the work studying (1.1)-(1.2) as a continuum model with the computational experiments using (1.1)-(1.2) by a numerical analysis of discretizations of (1.1)-(1.2). We thus consider stability and convergence of finite element discretizations of (1.1)-(1.2) as $h \to 0$. Our goal is to elucidate the interconnections between δ, h, χ, ν, and the algorithms used to compute the fluctuation $\mathbf{u}^*$ as a discrete function.

In Section 2 we give a precise definition of the discrete averaging operator and the de-convolution procedure that are used to define the generalized fluctuation $\mathbf{u}^*$. We also give preliminaries about the finite element discretizations studied. Section 3 gives the convergence analysis of this method. This analysis is for $\nu > 0$. The Euler equations, $\nu = 0$ in (1.1),(1.2), include shocks – a phenomenon excluded when $\nu > 0$. In Section 5 we complement the case $\nu > 0$ by considering a conservation law in one space dimension. We show that adding the time relaxation term $\chi\mathbf{u}^*$ does not alter shock speeds – thus confirming theoretically a result of Stoltz and Adams [1]. In Section 4 we give some numerical tests. Our primary goal in these tests is to study the effect the time relaxation term has on O(1) scales. We study a flow very close to its transition from one regime to another: from equilibrium to time dependent via eddy shedding behind the forward-backward step. We investigate experimentally which of several natural formulations of this time relaxation term least retards this transition.

2. Analysis of the Time Relaxation Model

In order to discuss the effects of the regularization we introduce the following notation. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $|\cdot|_{W_p^k}$. H^k is used to represent the Sobolev space W_2^k, and $\|\cdot\|_k$ denotes the norm in H^k. For functions $v(\mathbf{x},t)$ defined on the entire time interval $(0,T)$, we define

$$\|v\|_{\infty,k} := \sup_{0<t<T} \|v(t,\cdot)\|_k \,, \qquad \text{and} \qquad \|v\|_{m,k} := \left(\int_0^T \|v(t,\cdot)\|_k^m \, dt \right)^{1/m} .$$

The following function spaces are used in the analysis:

$$\begin{aligned} \text{Velocity Space} \quad &: \quad X := H_0^1(\Omega)\,, \\ \text{Pressure Space} \quad &: \quad P := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \,:\, \int_\Omega q \, d\Omega = 0 \right\} , \\ \text{Divergence}-\text{free Space} \quad &: \quad Z := \left\{ v \in X : \int_\Omega q \, \nabla\cdot\mathbf{v} \, d\Omega \,=\, 0, \ \forall\, q \in P \right\} . \end{aligned}$$

We denote the dual space of X as X', with norm $\|\cdot\|_{-1}$.

A variational solution of the N-S equations may be stated as: *Find* $\mathbf{w} \in L^2(0,T;X) \cap L^\infty(0,T;L^2(\Omega))$, $r \in L^2(0,T;P)$ *with* $\mathbf{w}_t \in L^2(0,T;X')$ *satisfying*

$$\begin{aligned} &(2.1) \quad (\mathbf{w}_t,\mathbf{v}) + (\mathbf{w}\cdot\nabla\mathbf{w},\mathbf{v}) - (r,\nabla\cdot\mathbf{v}) + \nu(\nabla\mathbf{w},\nabla\mathbf{v}) &=& \ (\mathbf{f},\mathbf{v}) \,,\ \forall \mathbf{v} \in X \,, \\ &(2.2) \quad (q,\nabla\cdot\mathbf{w}) &=& \ 0 \,,\ \forall q \in P \,, \\ &(2.3) \quad \mathbf{w}(0,\mathbf{x}) &=& \ \mathbf{w}_0(\mathbf{x}) \,,\ \forall \mathbf{x} \in \Omega \,. \end{aligned}$$