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DISCONTINUOUS GALERKIN APPROXIMATIONS FOR DISTRIBUTED OPTIMAL CONTROL PROBLEMS CONSTRAINED BY PARABOLIC PDE'S

KONSTANTINOS CHRYSAFINOS

This paper is dedicated to Prof. Max Gunzburger on the occasion of his 60th birthday

Abstract. A discontinuous Galerkin finite element method for optimal control problems having states constrained by linear parabolic PDE's is examined. The spacial operator may depend on time and need not be self-adjoint. The schemes considered here are discontinuous in time but conforming in space. Fully-discrete error estimates of arbitrary order are presented and various constants are tracked. In particular, the estimates are valid for small values of α, γ, where α denotes the penalty parameter of the cost functional and γ the coercivity constant. Finally, error estimates for the convection dominated convection-diffusion equation are presented, based on a Lagrangian moving mesh approach.

Key Words. Error estimates, discontinuous Galerkin, optimal control, parabolic PDE's, distributed control, convection dominated, convection-diffusion equations.

1. Introduction

The optimal control problem considered here is to minimize the functional,

$$J(y,u) = \frac{1}{2}\int_0^T \|y-z\|^2_{L^2(\Omega)}dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(\Omega)}dt \tag{1.1}$$

subject to the constraints,

$$\begin{cases} y_t + A(t)y &= u + g \qquad \text{in } (0,T)\times\Omega \\ y &= 0 \qquad\quad\;\; \text{on } (0,T)\times\Gamma \\ y(0,x) &= y_0 \qquad\quad \text{in } \Omega \end{cases} \tag{1.2}$$

where Ω denotes a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary Γ, y_0, g denote the initial data and the forcing term respectively, $A(t) : H^1_0(\Omega) \to H^{-1}(\Omega)$ is a time-dependent, possibly non-selfadjoint linear operator acting on Hilbert spaces through the standard pivot construction $H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. A typical example, is the operator $A(.) = -\text{div}[A_0(x,t)\nabla y]$ where $A_0(x,t)$ is a $C^1(\bar{\Omega})$, matrix-valued function that is uniformly positive definite. The distributed control variable is denoted by u, and α is a penalty parameter. The physical meaning of the optimization problem, is to seek states y and controls u such that y is close to the perscribed target z.

Even though the analysis of this problem is well understood, see e.g. [12, 28], several problems arise both in the analysis as well as in the implementation of efficient numerical algorithms (see e.g. [14]). In particular, the optimality system

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contains a (backward in time) adjoint equation which is coupled to the primal (forward in time) equation through an optimality condition and the regularity of solutions of the coupled system is typically low (see, e.g [12, 14, 28]). The choice of the $L^2[0,T;L^2(\Omega)]$ norm at the functional leads to an algebraic optimality condition which greatly simplifies the implementation of the numerical algorithm. However, in many computational examples (see also e.g. [14] for relevant discussions for the velocity tracking problem), the size of the parameter α is very important. In general, small values of α allow bigger values of the control function u, which implies faster convergence of the state variable y to the specified target z. Furthermore, in many interesting physical applications, such as (convection dominated) convection-diffusion problems the values of the coercivity constant related to the operator $A(.)$ are typically low, and comparable to the mesh size h. Therefore, the penalty parameter α as well as the coercivity constant γ should be tracked.

The scope of this work is the analysis of a classical discontinuous Galerkin scheme which is discontinuous in time and conforming in space and the derivation of fully-discrete a-priori error estimates. It is well known that the discontinuous Galerkin method performs very well in vast area of problems where the given data satisfy low regularity properties, such as optimal control problems, while it accomodates the use of different subspaces in each time step, and hence basic adaptivity ideas, in a natural way. This work can be viewed as an extension of various ideas and techniques of [3] related to the discontinuous Galerkin approximation of parabolic equations, in the case of constrained optimization problems. It is also motivated by the recent work of [30], where a-posteriori error estimates for a discontinuous Galerkin scheme are derived for an optimal control problem.

The main focus of our estimates is to fully utilize these favourable properties of discontinuous Galerkin methods in the context of optimal control problems. In particular the discontinuous Galerkin scheme analyzed here, exhibits the following features:

- Facilitates the uncoupling of the state and adjoint variables, and leads to estimates that are valid under minimal regularity assumptions on the given data.
- Various constants such as the continuity and coercivity constants β, γ respectively and the penalty parameter α are carefully tracked and do not appear at any exponential.
- The estimates are stated in terms of various projections and are applicable in case of approximation based on high order finite element spaces, provided that natural regularity and compatibility assumptions on the given data are valid.
- The operator $A(t)$ may depend on time and need not be self-adjoint.

To our best knowledge in the context of constrained optimal control problems, these estimates are new. The rest of paper is organized as follows. In the remaining of this section, we present some related results (see also references in [14, 28, 32]), while in section 2 we formulate the continuous optimal control problem. In section 3, the main estimates are presented in the energy norms as well as at the partition points, followed by estimates at arbitrary points. Finally, a Lagrangian moving mesh approach similar to [4] is described for the convection dominated convection-diffusion equation.

1.1. Related results. The literature regarding optimal control problems constrained to time-dependent equations is vast. Several problems with distributed