INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 4, Number 3-4, Pages 713–728

ANALYSIS AND APPROXIMATIONS OF A TERMINAL-STATE OPTIMAL CONTROL PROBLEM CONSTRAINED BY SEMILINEAR PARABOLIC PDES

L. S. HOU AND H.-D. KWON

This paper is dedicated to Professor Max Gunzburger on the occasion of his 60th birthday.

Abstract. A terminal-state optimal control problem for semilinear parabolic equations is studied in this paper. The control objective is to track a desired terminal state and the control is of the distributed type. A distinctive feature of this work is that the controlled state and the target state are allowed to have nonmatching boundary conditions. The existence of an optimal control solution is proved. We also show that the optimal solution depending on a parameter γ gives solutions to the approximate controllability problem as $\gamma \to 0$. Error estimates are obtained for semidiscrete (spatially discrete) approximations of the optimal control problem. A gradient algorithm is discussed and numerical results are presented.

Key Words. Terminal-state tracking, optimal control, semilinear parabolic equations, approximate controllability.

1. Introduction

In this paper we study a terminal-state tracking optimal control problem for a semilinear second order parabolic partial differential equation (PDE) defined over a finite time horizon $[0,T] \subset [0,\infty)$ and on a bounded, C^2 (or convex) spatial domain $\Omega \subset \mathrm{R}^d$, $d = 1$ or 2 or 3. Let $W \in L^2(\Omega)$ denote a given target function, $w \in L^2(\Omega)$ a given initial condition. Let $f \in L^2((0,T)\times\Omega)$ denote the distributed control. We wish to find a control f that drives the state u to W at time T, and we will use the optimal control approach to achieve this objective. The optimal control problem modeling such an objective is to minimize the terminal-state tracking functional

$$\mathcal{J}(u,f) = \frac{T}{2}\int_\Omega |u(T,\mathbf{x}) - W(\mathbf{x})|^2\,d\mathbf{x} + \frac{\gamma}{2}\int_0^T\int_\Omega |f(t,\mathbf{x}) - F(t,\mathbf{x})|^2\,d\mathbf{x}\,dt \tag{1.1}$$

(where γ is a given positive constant and F a given reference function) subject to the parabolic PDE

$$\partial_t u - \mathrm{div}\,[A(\mathbf{x})\nabla u] + \Phi(u) + a(\mathbf{x})u = f, \qquad (t,\mathbf{x}) \in (0,T)\times\Omega \tag{1.2}$$

with the homogeneous boundary condition

$$u = 0, \qquad (t,\mathbf{x}) \in (0,T)\times\partial\Omega \tag{1.3}$$

and the initial condition

$$u(0,\mathbf{x}) = w(\mathbf{x}), \qquad \mathbf{x} \in \Omega\,. \tag{1.4}$$

Received by the editors March 10, 2006.

2000 *Mathematics Subject Classification.* 93B05, 93B06, 93B40, 93C20, 93C50, 65M60.

In (1.2), $A(\mathbf{x})$ is a symmetric matrix-valued, $C^1(\overline{\Omega})$ function that is uniformly positive definite and

$$a(\mathbf{x}) \in L^\infty(\Omega), \quad a(\mathbf{x}) \geq -C_1 \tag{1.5}$$

where C_1 is a positive constant. We assume that the function $\Phi(u) \in C^1(I\!R)$ satisfies the conditions

$$\Phi'(u) \geq 0 \quad \text{and} \quad M_1|u|^{p_0} + M_2|u| \geq \Phi(u)\cdot u \geq \mu_0|u|^{p_0} \tag{1.6}$$

where M_1, M_2, $\mu_0 > 0$ and $p_0 > 2$.

Optimal control problems of similar type for *linear* parabolic equations can be found in the literature, e.g., [4, 13, 16]. Just as in [16], a distinctive feature of this work is that the desired terminal-state W and the admissible state u are allowed to have nonmatching boundary conditions, though the reference function F needs be suitably chosen in the formulation of cost functional (1.1) (the details about the choice of F will be revealed in Section 2). But unlike [16], eigen series methods will not work in the present nonlinear setting. The techniques developed in the study of like problems for the Navier-Stokes equations [17] turn out to be useful for the setting herein.

Terminal-state tracking problems are optimal control problems in their own right. They are also closely related to approximate and exact controllability problems in linear or nonlinear settings which were studied in, among others, [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 20, 21]. We shall establish an approximate controllability result for the underlying nonlinear PDE.

This paper is organized as follows. In Section 2 we formulate the semilinear optimal control problem in an appropriate mathematical framework, depending on a parameter γ. In Section 3 we prove the existence of an optimal solution u_γ. In Section 4 we show that the optimal solution u_γ at the terminal time T approaches the target state W as the parameter $\gamma \to 0$, i.e. the optimal solution is a solution of approximate controllability problem. In Section 5 we discretize the spatial variables by finite element methods and consider dynamics of the semidiscrete optimal solution. In Section 6 we introduce a two-dimensional algorithm based on the gradient method to compute the optimal solution; we also will present some computational results.

2. Formulation of optimal control problem

As outlined in Section 1, we study in this paper the semilinear optimal control problem: minimize the terminal-state tracking functional (1.1) subject to the semilinear parabolic equations (1.2)–(1.4). Throughout we freely make use of standard Sobolev space notations $H^m(\Omega)$ and $H^1_0(\Omega)$. We denote the norm for Sobolev space $H^m(\Omega)$ by $\|\cdot\|_m$. Note that $H^0(\Omega) = L^2(\Omega)$ so that $\|\cdot\|_0$ is the $L^2(\Omega)$ norm. We will need the temporal-spatial function space

$$H^{2,1}(Q) = \{v \in L^2(0,T;H^2(\Omega)) : v_t \in L^2(0,T;L^2(\Omega))\}$$

where

$$Q = (0,T)\times\Omega\,.$$

A temporal-spatial function $v(t,\mathbf{x})$ often will be simply written as $v(t)$.

We construct the reference function F in (1.1) as follows. We first choose a one-parameter set of functions $\{W^{(\gamma)} : \gamma > 0\} \subset H^2(\Omega) \cap H^1_0(\Omega) \cap L^{p_0}(\Omega)$ such that

$$\|W^{(\gamma)} - W\|_0 \to 0 \quad \text{as } \gamma \to 0\,. \tag{2.1}$$