INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 1, Pages 112–131

EXTRAPOLATION FOR THE SECOND ORDER ELLIPTIC PROBLEMS BY MIXED FINITE ELEMENT METHODS IN THREE DIMENSIONS

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Abstract. In this paper we derive asymptotic error expansions for mixed finite element approximations of general second order elliptic problems in three dimensions. And extrapolation method is applied to improve the accuracy of the approximations with the help of the interpolation postprocessing technique. For the cubic domain and uniform partition, with the extrapolation, the accuracy of the mixed finite element approximations can be improved.

Key Words. Superconvergence, interpolation postprocessing, extrapolation, mixed finite element

1. Introduction

We are concerned with the approximations of the following system:

$$\left\{\begin{array}{rcll} \mathbf{u}+A\nabla p &=& 0 & \text{in } \Omega, \\ \nabla\cdot\mathbf{u}+cp &=& f & \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} &=& 0 & \text{on } \Gamma, \end{array}\right. \tag{1.1}$$

where ∇ and $\nabla\cdot$ are the gradient and divergence operators, $\Omega \subset \mathbf{R}^3$ is an open bounded cubic domain with boundary Γ, $\mathbf{n}$ indicates the outward unit normal vector along Γ, $A^{-1} = (\alpha_{ij})_{3\times 3}$ is a full positive definite matrix uniformly in Ω.

Mixed finite element methods [1] should be employed to discretize the system (1.1).

The main content of this paper is to present an analysis for the extrapolation of the mixed finite elements in three dimensions. The application of this approach in finite element methods was first established by Q. Lin [12]. The extrapolation method relies heavily on the existence of an asymptotic expansion for the error. The extrapolation of mixed finite element approximation in two dimensions was studied in [5]. In this paper, we study the three dimensional case.

This paper is organized in the following way. In Section 2, we establish the approximation subspace and the variational formulation for the problem (1.1) and the Raviart-Thomas interpolation. The asymptotic expansion for the Raviart-Thomas interpolation is derived in Section 3. Section 4 is devoted to investigating the asymptotic expansion of the error between the mixed finite element solution and the Raviart-Thomas interpolation of the exact solution to (1.1). Based on the expansion, the asymptotic expansion of the mixed finite element approximation is demonstrated by an interpolation postprocessing method in Section 5. Hence, The extrapolation can be used to improve the accuracy of the mixed finite element solution. Some concluding remarks are given in the finial section.

Received by the editors September 10, 2005 and, in revised form, December 22, 2006.
2000 *Mathematics Subject Classification.* 65N30, 65M60, 31A30.

2. The mixed finite element method and Raviart-Thomas interpolation

In this section, we formulate the mixed finite element method for the second order elliptic differential equation (1.1).

Let

$$W := L^2(\Omega), \quad \mathbf{V} := \mathbf{H}(\mathrm{div}, \Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^3 : \ \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}$$

be the standard L^2 space on Ω with the norm $||\cdot||_0$ and the Hilbert space equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{V}} := \left(\|\mathbf{v}\|_0^2 + \|\nabla \cdot \mathbf{v}\|_0^2 \right)^{\frac{1}{2}},$$

respectively. In addition, set

$$\mathbf{V}_0 := \{ \mathbf{v} \in \mathbf{V} : \mathbf{v} \cdot \mathbf{n} = 0, x \in \Gamma \}.$$

So, the corresponding weak mixed formulation for (1.1) seeks $(\mathbf{u}, p) \in \mathbf{V}_0 \times W$ such that

$$a(\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_0, \tag{2.1}$$

$$(\nabla \cdot \mathbf{u}, \omega) + (cp, \omega) = (f, \omega) \quad \forall \omega \in W, \tag{2.2}$$

where $a(\cdot, \cdot)$ is a bilinear form defined by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega A^{-1} \mathbf{u}\mathbf{v} dxdydz$$

and $(\cdot, \cdot)$ denotes the standard L^2-inner product.

Let $\mathbf{T}_{h_1,h_2,h_3}$ be a finite element partition of Ω into uniform hexahedrons, $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote a pair of finite element spaces satisfying the LBB condition, h_1, h_2 and h_3 denote the mesh sizes in $x-$,$y-$ and $z-$ axis and

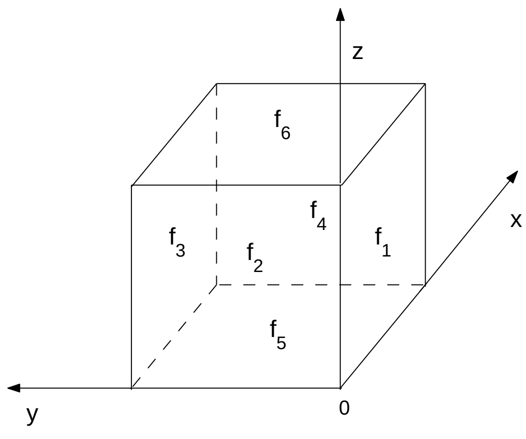
$$\mathbf{V}_h := \{ \mathbf{v}_h \in \mathbf{V}; \mathbf{v}_h|_e \in Q_{100} \times Q_{010} \times Q_{001}, \forall e \in \mathbf{T}_{h_1,h_2,h_3} \},$$
$$W_h := \{ w_h \in W : W_h|_e \in Q_{000}, \forall e \in \mathbf{T}_{h_1,h_2,h_3} \},$$

where Q_{ijk} denotes the space of polynomials of degree no more than i, j and k in x, y and z direction, respectively.

Let $\mathbf{V}_{0h} := \{ v \in \mathbf{V}_h : \mathbf{v} \cdot \mathbf{n} = 0, x \in \Gamma \}$. Hence, the corresponding discrete mixed finite element version of (1.1) is defined to seek $(\mathbf{u}_h, p_h) \in \mathbf{V}_{0h} \times W_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.3}$$

$$(\nabla \cdot \mathbf{u}_h, \omega) + (cp_h, \omega) = (f, \omega) \quad \forall \omega \in W_h. \tag{2.4}$$



The hexahedron element e and its six faces