INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 1, Pages 132–151

FINITE VOLUME ELEMENT METHOD FOR SECOND ORDER HYPERBOLIC EQUATIONS

SARVESH KUMAR, NEELA NATARAJ AND AMIYA K. PANI

Abstract. We discuss a priori error estimates for a semidiscrete piecewise linear finite volume element (FVE) approximation to a second order wave equation in a two-dimensional convex polygonal domain. Since the domain is convex polygonal, a special attention has been paid to the limited regularity of the exact solution. Optimal error estimates in L^2, H^1 norms and quasioptimal estimates in L^∞ norm are discussed without quadrature and also with numerical quadrature. Numerical results confirm the theoretical order of convergence.

Key Words. finite element, finite volume element, second order hyperbolic equation, semidiscrete method, numerical quadrature, Ritz projection, optimal error estimates.

1. Introduction

In this paper, we are interested in the finite volume element method (FVEM) for the following second order linear hyperbolic initial boundary value problem :
Given $f(x,t)$, $g(x)$ and $w(x)$, and $t \in (0,T]$ for $x \in \Omega$, find $u = u(x,t)$ such that

$$\begin{aligned} u_{tt} - \nabla.(A(x)\nabla u) &= f(x,t) \quad \forall x \in \Omega,\ 0 < t \leq T, \\ u(x,t) &= 0 \qquad\quad \forall x \in \partial\Omega,\ 0 < t \leq T, \\ u(x,0) &= g(x) \qquad \forall x \in \Omega, \\ u_t(x,0) &= w(x) \qquad \forall x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded, convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $A(x) = (a_{ij}(x))_{i,j=1}^2$ is a real-valued and uniformly positive definite matrix in Ω. It is assumed that the functions $f,\ g,\ w$ have enough regularity and they satisfy appropriate compatibility conditions so that the boundary value problem (1.1) has a unique solution satisfying the regularity results as demanded by our subsequent error analysis.
The FVEM employs a finite element partition of the domain $\overline{\Omega} = \Omega \cup \partial\Omega$. It may be considered as a Petrov-Galerkin finite element method in which the trial space is C^0- piecewise linear on the finite element partition of $\overline{\Omega}$ and the test space is piecewise constant over the control volume to be defined in Section 2. The FVEM has been studied by Bank and Rose [3], Cai [4], Chatzipantelidis [6], R. Li et al. [13], Ewing et al. [10], etc. for elliptic problems and by Chou et al. [5], Chatzipantelidis et al. [7] and Sinha et al. [18] for parabolic problems. For elliptic problems, the authors [13] have obtained optimal order H^1 and L^2 error estimates of the following form

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_{W^{3,p}(\Omega)},\ p > 1,$$

Received by the editors July 20, 2006.

2000 *Mathematics Subject Classification.* 65N30, 65N15.

where u is the exact solution and u_h is the FV approximation of u. Note that the regularity on the exact solution seems to be too high compared to that for the finite element methods. On the other hand, it may be difficult to have $u \in H^3$ if Ω is a convex polygonal domain. The authors [10] have derived optimal L^2 error estimate assuming that the exact solution $u \in H^2$ and the source term $f \in H^1$. They have also provided an example that if $f \in L^2$, then FVE solution may not have optimal error estimates in L^2 norm. In [7], the authors have extended the analysis of [10] to parabolic problems in a convex polygonal domain. They have also considered the effect of quadrature for the L^2 inner product and derived a priori error estimates. Ewing et al. [11] have discussed a priori error estimates for the parabolic integro-differential equations using FVEM. In the present paper, we have extended the results to include the second order hyperbolic equation. Special attention has been paid for the data with minimal regularity assumption. Moreover, the effect of quadrature is also discussed.

Let us relate our work with the literature for the second order hyperbolic equations. R. Li et al. [13] have proved the optimal order of convergence in H^1- norm without quadrature using elliptic projection, but the regularity of the exact solution seems to be high compared to our results. The finite element analysis for the second order hyperbolic equations without quadrature was discussed by Baker [1] and with quadrature by Baker and Dougalis [2] and Dupont [9]. Baker and Dougalis [2] have proved that the finite element solution for hyperbolic equation has optimal order convergence in $L^\infty(L^2)$ for the semidiscrete scheme, provided $g \in H^5 \cap H_0^1$ and $w \in H^4 \cap H_0^1$. In [15], Rauch has also discussed the convergence of the Galerkin approximation to a second order wave equation by using piecewise linear polynomials and proved optimal $L^\infty(L^2)$ estimate with $g \in H^3 \cap H_0^1$ and $w = 0$ which are the minimal regularity conditions for the second order wave equation. Pani et al. [14] and Sinha [17] have also studied the effect of numerical quadrature in finite element method for parabolic and hyperbolic integro-differential equations with the assumption that $g \in H^3 \cap H_0^1$ and $w \in H^2 \cap H_0^1$. In this paper, we have derived optimal $L^\infty(L^2)$ estimate even with quadrature when $g \in H^3 \cap H_0^1$ and $w \in H^2 \cap H_0^1$.

This paper is organized as follows: In Section 3, optimal order of convergence in L^2 and H^1 norms for the semidiscrete scheme without quadrature and with the assumption that the initial functions g, w are in $H^3 \cap H_0^1$ and $H^2 \cap H_0^1$, respectively, has been derived. Moreover, quasi-optimal order of convergence in maximum norm has also been proved. The integrals occurring in the semidiscrete scheme are replaced by quadrature formulae. The effect of numerical quadrature on the estimates has been discussed in Section 4. The analysis is based on the properties of the standard Ritz projection. In both Sections 3 and 4, the error estimates are derived under the assumption that the domain is convex polygon. In order to verify the derived order of convergence, some numerical experiments are discussed in Section 5.

2. Notation and Preliminaries.

In this paper, we use the standard notation for the Sobolev spaces. Let $W^{s,p}(\Omega)$ with $1 \le p \le \infty$ consist of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\sum_{|\alpha| \le s} \|D^\alpha u\|_{L^p}^p \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$