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NUMERICAL CHARACTERIZATION OF THE REGULARITY LOSS IN MINIMAL SURFACES

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Abstract. In this article, we numerically study the regularity loss of the solutions of non-parametric minimal surfaces with non-zero boundary conditions. Parts of the boundaries have non-positive mean curvature. As expected from theoretical results in such geometry, we find that the solutions may or may not satisfy the boundary conditions depending upon the data. Firstly, we validate the numerical study on the astroid and discuss the various kinds of non-regularity characterizations. We provide an algorithm to test the regularity loss using the numerical results. Secondly, we give a numerical estimate of the threshold value of the boundary condition beyond which no regular solution exists. More theoretical results are also given on the approximation by the regularized solution of the non-regularized one. The regularized solution exhibits a boundary layer. Finally, the study is applied to the catenoid for which the exact threshold value is known. The exact value and the computed one are in good agreement.

Key Words. Boundary layers, minimal surfaces, non-regular solutions, singular perturbations.

1. Introduction

1.1. The context. Minimal surfaces have received attention at least since the publication of L. Euler's book [6] in 1744. In this book, L. Euler discussed one hundred problems, one of them being to find surfaces of revolution having a critical area. Later, J.L. Lagrange made the connection between vanishing mean curvature and the first variation of area [13]. From experiments, J.A.F. Plateau noticed that the singular set Γ of a soap bubble cluster is a piecewise smooth curve with vertices, satisfying two laws [14]. These laws were proved only in 1976 with the theory of varifolds by F. Almgren [1] and the regularity theorem of J. Taylor [19].

Almost two centuries and a half later, this topic is still active. In the Summer of 2001, the Mathematical Sciences Research Institute (MSRI) hosted the Clay Mathematics Institute Summer School on the Global Theory of Minimal Surfaces. The nature of the meeting made it possible to give a panoramic view of this subject. The subjects covered include minimal and constant-mean-curvature submanifolds, geometric measure theory and the double-bubble conjecture, Lagrangian geometry, numerical simulation of geometric phenomena, applications of mean curvature

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to general relativity and Riemannian geometry, the isoperimetric problem, the geometry of fully nonlinear elliptic equations and applications to the topology of three-dimensional manifolds. An edited book was published in 2005 [10].

The introduction of computer graphics enabled many numerical studies (see for instance [15]). Some of these studies made possible the proof of theoretical results such as the one of Hoffman and Meeks [11].

The problem of non-parametric minimal surfaces consists in finding a graph function u solution of the following minimization functional :

$$\min_{u|_\Gamma=\Phi} \int_\Omega \sqrt{1+|\nabla u|^2}\, dx, \tag{1}$$

for u defined inside Ω, Γ being the boundary of Ω on which Φ is a given L^∞ (or smoother) function.

We recall here that the Euler-Lagrange equation corresponding to (1) and the associated boundary condition read as follows :

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = 0, \quad \text{in } \Omega, \tag{2}$$

$$u = \Phi, \quad \text{on } \Gamma. \tag{3}$$

Many works were concerned with the existence of a strong solution for (1) in the case of boundary of non-negative curvature, such as [5], and many others. Since the 1970s, several papers proved the existence of weak solutions of (1) called "generalized solutions" satisfying either $u|_{\Gamma'} = \Phi$ or $\partial u/\partial n(\Gamma') = \infty$, for $\Gamma' \subset \Gamma$: see e.g. [3] and [20] and see also [12] for relevant a priori estimates in a more general context. So the generalized solution develops a "vertical branch" and its normal derivative becomes infinite near the boundary in the region where $u = \Phi$ is not satisfied. Of course here the minimal surface is non-parametric. These results were obtained by two different arguments. On the one hand, Bombieri, De Giorgi and Miranda [3] and Giusti [8] obtained a generalized solution by a purely geometric argument. On the other hand, Temam used in [20] a duality argument and defined also a generalized solution as the limit of sequences of some regularized solutions in the following manner :

$$\min_{u^\varepsilon|_\Gamma=\Phi} \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \sqrt{1+|\nabla u^\varepsilon|^2}\Big) dx, \tag{4}$$

i.e.,

$$-\varepsilon\Delta u^\varepsilon - \operatorname{div}\Big(\frac{\nabla u^\varepsilon}{\sqrt{1+|\nabla u^\varepsilon|^2}}\Big) = 0, \quad \text{in } \Omega, \tag{5}$$

$$u^\varepsilon = \Phi, \quad \text{on } \Gamma. \tag{6}$$

Here ε is a small non-negative parameter which may tend to zero. The term $-\varepsilon\Delta u^\varepsilon$ in (5) constitutes an elliptic regularization making the functional in (4) strictly convex and the corresponding Euler equation (5) uniformly elliptic. So, we have by classical theorems the existence and uniqueness of a regular solution ($\mathcal{C}^2(\overline{\omega}) \quad \forall\, \omega \subset\subset \Omega$ [1]). It is obvious that this fails to be true up to the boundary

[1] $\omega \subset\subset \Omega$ means $\omega \subset \overline{\omega} \subset \Omega$ as Ω is a bounded open set.