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A UNIFORMLY CONVERGENT METHOD ON ARBITRARY MESHES FOR A SEMILINEAR CONVECTION-DIFFUSION PROBLEM WITH DISCONTINUOUS DATA

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Abstract. This paper deals with a uniform (in a perturbation parameter) convergent difference scheme for solving a nonlinear singularly perturbed two-point boundary value problem with discontinuous data of a convection-diffusion type. Construction of the difference scheme is based on locally exact schemes or on local Green's functions. Uniform convergence with first order of the proposed difference scheme on arbitrary meshes is proven. A monotone iterative method, which is based on the method of upper and lower solutions, is applied to computing the nonlinear difference scheme. Numerical experiments are presented.

Key Words. convection-diffusion problem, discontinuous data, boundary layer, uniform convergence, monotone iterative method.

1. Introduction

We are interested in the semilinear two-point boundary-value problem with a convective dominated term and discontinuous data

$$-\varepsilon u'' + b(x)u' + c(x,u) + f(x) = 0, \quad x \in \omega = (0,1), \tag{1}$$

$$u(0) = 0,\ u(1) = 0, \quad b(x) \geq b_* = \text{const} > 0,\ c_u \geq 0,\ (c_u \equiv \partial c/\partial u),$$

where ε is a small positive parameter. Suppose that the function c is sufficiently smooth and b, f are piecewise smooth functions, i.e.

$$b(x), f(x) \in Q_p^n(\overline{\omega}),\ n \geq 0.$$

We say that $v(x) \in Q_p^n(\overline{\omega})$ if it is defined on $\overline{\omega}$ and has derivatives up to order n, the function itself and its derivatives may only have jump discontinuities at a finite set of points $p = \{p_1, \ldots, p_J\}, 0 < p_j < p_{j+1}, j = 1, \ldots, J-1$, i.e. $Q_p^n(\overline{\omega}) = C^n(\overline{\omega} \backslash p)$.

The solution to (1) is a function with a continuous first derivative, which satisfies the boundary conditions and the equation everywhere, with the exception of the points in p. The problem (1) has a unique solution [9]

$$u(x) \in C^1(\overline{\omega}) \cap Q_p^{n+2}(\overline{\omega}).$$

Linear versions of problem (1) with discontinuous data are investigated in [2], [5]. The solution of the linear problem possesses a strong boundary layer at $x = 1$ and weak interior layers at the points of discontinuity p. The boundary layer is strong in the sense that the solution is bounded, but the magnitude of its first derivative at $x = 1$, grows unboundedly as $\varepsilon \to 0$. The interior layers at p are weak: i.e.

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the solution and the first derivative are bounded but the magnitude of the second derivative grows unboundedly as $\varepsilon \to 0$. We show (see Lemma 1) that problem (1) possesses a strong boundary layer at $x = 1$ and the solution and the first derivative are bounded at the points of discontinuity p.

Our goal is to construct an ε-uniform numerical method for solving problem (1), that is, a numerical method which generates ε-uniformly convergent numerical approximations to the solution. In [2], [5], for solving the linear version of problem (1), the uniform numerical methods are constructed by using the integral-difference method (or the method of locally exact schemes) on arbitrary nonuniform meshes [2], and by using the standard upwind finite difference method on the piecewise uniform mesh, which is fitted to boundary and interior layers [5].

In the next section, we establish some a priori estimates of the solution and its first derivative. In Section 3 we construct a numerical method by applying the integral-difference approach. Note that in the constructed numerical method, a difference operator corresponding to the linear differential operator $-\varepsilon d^2/dx^2 + bd/dx$ is equivalent to the upwind finite volume method from [6], [10]. In Section 4 we prove uniform convergence of the numerical method on arbitrary nonuniform meshes by extending in a natural way the proof of the main theoretical result from [3] (the difference scheme in the case of problem (1) with smooth data converges ε-uniformly). In Section 5 we construct a monotone iterative method for solving the nonlinear difference scheme and prove that the iterates converge ε-uniformly to the solution of problem (1). In the last section numerical results are presented, which are in agreement with the theoretical results.

2. Properties of the continuous problem

The following lemma contains *a priori* estimates of the solution to problem (1).

Lemma 1. *If $b(x), f(x) \in Q_p^n(\overline{\omega})$, $n \geq 0$, then a unique solution to (1) exists and $u(x) \in C^1(\overline{\omega}) \cap Q_p^{n+2}(\overline{\omega})$. The solution $u(x)$ satisfies the following estimates:*

$$\left| \frac{d^k u(x)}{dx^k} \right| \leq C \left[1 + \varepsilon^{-k} \exp\left(-\frac{b_*(1-x)}{\varepsilon} \right) \right], \ x \in \overline{\omega}, \quad k = 0, 1,$$

here and throughout the paper, C denotes a generic positive constant independent of ε.

Proof. The result that problem (1) with the piecewise smooth functions b and f has a unique solution can be found in [9].

Firstly, we estimate the solution $u(x)$ to (1). The transformation $u(x) = e^{\gamma x} w(x)$ with a positive constant γ yields the equation and the boundary conditions

$$-\varepsilon w^{''} + \tilde{b}(x) w^{'} + \tilde{c}(x, w) + e^{-\gamma x} f = 0, \quad w(0) = w(1) = 0,$$

$$\tilde{b} = b - 2\varepsilon\gamma, \quad \tilde{c}(x, w) = e^{-\gamma x} c(x, e^{\gamma x} w) + (b\gamma - \varepsilon\gamma^2) w.$$

If we choose $\gamma = b_*/4$ and assume that $\varepsilon \leq 1$, then

$$\tilde{b}(x) \geq \tilde{b}_* = b_*/2, \quad \tilde{c}_w \geq \tilde{c}_* = (3/16) b_*^2.$$

If $w(x)$ is the exact solution of the above problem, then by the mean-value theorem, we can represent $\tilde{c}(x, w)$ in the form

$$\tilde{c}(x, w) = \tilde{c}(x, 0) + \tilde{c}_w(x) w(x),$$

where $\tilde{c}_w(x) = \tilde{c}_w \left(x, \theta(x) w(x) \right)$, $0 < \theta(x) < 1$. Assuming that $\tilde{c}_w(x)$ is given as a function of x, then the solution $w(x)$ may be considered as a solution of the linear