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A NEW HIGH ORDER TWO LEVEL IMPLICIT DISCRETIZATION FOR THE SOLUTION OF 3D NON-LINEAR PARABOLIC EQUATIONS

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Abstract. We present a new two-level implicit difference method of $O(k^2 + kh^2 + h^4)$ for approximating the three space dimensional non-linear parabolic differential equation $u_{xx}+u_{yy}+u_{zz} = f(x, y, z, t, u, u_x, u_y, u_z, u_t), 0 < x, y, z < 1, t > 0$ subject to appropriate initial and Dirichlet boundary conditions, where $h > 0$ and $k > 0$ are mesh sizes in space and time directions, respectively. In addition, we also propose some new two-level explicit stable methods of $O(kh^2+h^4)$ for the estimates of $(\partial u/\partial n)$. When grid lines are parallel to $x-, y-$ and $z-$ coordinate axes, then $(\partial u/\partial n)$ at an internal grid point becomes $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$, respectively. In all cases, we require only 19-spatial grid points and a single computational cell. The proposed methods are directly applicable to singular problems and we do not require any special technique to handle singular problems. We also discuss operator splitting method for solving linear parabolic equation. This method permits multiple use of the one-dimensional tri-diagonal solver. It is shown that the operator splitting method is unconditionally stable. Numerical tests are conducted which demonstrate the accuracy and effectiveness of the methods developed.

Key Words. non-linear parabolic equation, implicit scheme, high order method, normal derivatives, singular problem, operator splitting, Burgers' equation.

1. Introduction

Three space dimensional non-linear parabolic partial differential equations represent mathematical models of physical problems of great interest in physics and applied mathematics. Numerical solution of three space dimensional parabolic problems tends to be computationally intensive and may be prohibitive on conventional computers due to the requirements on the memory and the CPU time to obtain solutions of required accuracy. Traditional numerical methods are of lower order and require extremely smaller grid lengths. The size of the resulting linear or non-linear systems for 3-space dimensional problem is usually so large that even present day computers may not be able to handle them. One approach to alleviate these difficulties is to use higher-order methods, which yield approximates solutions with comparable accuracy using much coarser discretization, resulting in linear or non-linear systems of smaller size. It has been repeatedly demonstrated on model problems that even the simplest types of high-order methods should provide tremendous practical advantages in terms of diminishing the required number of storages and also the overall computing time for a desired solution (see Ciment et al [1]). Several authors have discussed high order finite difference methods for the

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solution of three space dimensional linear parabolic equations (see Ciment et al [1], Iyengar and Manohar [2] , Zhang and Zhao [3]). The solution requires the inversion of a block banded matrix. Alternating direction implicit (ADI) methods originally developed for a two-space dimensional diffusion equation have been extended to three-space dimension by Douglas and Rochford [4] , Brian [5] and Fairweather and Mitchell [6, 7]. Two-level implicit difference methods of order 2 in time and 4 in space for the numerical solution of three-space dimensional non-linear parabolic equations have been discussed by Jain et al [8], Mohanty and Jain [9], Mohanty [10] and Mohanty et al [11]. However, their methods are not directly applicable to singular parabolic problems. A special technique is required to handle singular parabolic problems.

In this paper, we consider the numerical solution of the non-linear parabolic partial differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f(x,y,z,t,u,u_x,u_y,u_z,u_t), 0<x,y,z<1, t>0 \tag{1}$$

where $u=u(x,y,z,t)$. Let $\Omega=\{(x,y,z,t)|0<x,y,z<1,t>0\}$ be our solution domain with boundary $\partial\Omega$.

The initial condition is given by

$$u(x,y,z,0)=u_0(x,y,z) \qquad 0\le x,y,z\le 1 \tag{2}$$

and the boundary conditions are given by

$$u(0,y,z,t)=g_0(y,z,t), u(1,y,z,t)=g_1(y,z,t), 0\le y,z\le 1, t\ge 0 \tag{3}$$

$$u(x,0,z,t)=h_0(x,z,t), u(x,1,z,t)=h_1(x,z,t), 0\le x,z\le 1, t\ge 0 \tag{4}$$

$$u(x,y,0,t)=i_0(x,y,t), u(x,y,1,t)=i_1(x,y,t), 0\le x,y\le 1, t\ge 0 \tag{5}$$

where $u_0, g_0, g_1, h_0, h_1, i_0, i_1$ are given functions of sufficient smoothness.

In this paper, using 19-spatial grid points and a single computational cell (see Figure 1) we propose new formulas of order 2 in time and 4 in space coordinates for the solution of non-linear parabolic equation (1) and the estimates of $(\partial u/\partial n)$. When grid lines are parallel to coordinate axes, $(\partial u/\partial n)$ represents $(\partial u/\partial x)$, $(\partial u/\partial y)$ and $(\partial u/\partial z)$, respectively. The proposed methods are directly applicable to singular parabolic equations. We do not require any special technique or modification to handle the singular problem. Recently, Mohanty and Singh [12] have proposed a new fourth order finite difference method for the solution of three dimensional singularly perturbed non-linear elliptic partial differential equation. In next section, we give the description of new algorithms. The complete derivation of numerical methods is given in Section 3. In section 4, we discuss operator splitting method for the solution of a linear three space dimensional parabolic equation and its stability analysis. The operator splitting method requires the solution of tri-diagonal system of equations parallel to coordinates axes, at each time step, independent of the order of the method. In section 5, computational results of some test problems are provided to demonstrate the accuracy of the proposed numerical methods and compared with the corresponding second order methods. It is shown here that for a fixed mesh ratio parameter, the proposed methods are of fourth order in space. Concluding remarks are given in section 6.

2. Description of numerical algorithms

As usual, let us assume that the solution domain Ω is covered by a set of cubic grid with spacing $h>0$ and $k>0$ in space and time coordinates, respectively. The grid points (x_l, y_m, z_n, t_j) are given by $x_l=lh, y_m=mh, z_n=nh, t_j=jk$,