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HOW RATES OF L^p-CONVERGENCE CARRY OVER TO NUMERICAL APPROXIMATIONS OF SOME CONVEX, NON-SMOOTH FUNCTIONALS OF SDES

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Abstract. The relation between weak and p-th mean convergence of numerical methods for integration of some convex, non-smooth and path-dependent functionals of ordinary stochastic differential equations (SDEs) is discussed. In particular, we answer how rates of p-th mean convergence carry over to rates of weak convergence for such functionals of SDEs in general. Assertions of this type are important for the choice of approximation schemes for discounted price functionals in dynamic asset pricing as met in mathematical finance and other commonly met functionals such as passage times in engineering.

Key Words. stochastic differential equations, approximation of convex and path-dependent functionals, numerical methods, stability, L^p-convergence, weak convergence, rates of convergence, non-negativity, discounted price functionals, asset pricing, approximation of stochastic exponentials

1. Introduction

Suppose that the risky asset price $(X(t))_{t\geq 0}$ is governed by systems of Itô-type stochastic differential equations (SDEs) such as noisy ordinary differential equations

$$dX(t) = a(t, X(t))dt + \sum_{j=1}^{m} b^j(t, X(t))dW_j(t) \tag{1}$$

driven by Wiener processes or martingale-type noises W_j with respect to the forward filtration $(\mathcal{F}_t)_{t\geq 0}$ on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For some overview on the theory of SDEs, e.g. see Arnold [2], Gard [10], Oksendal [20] or Protter [21].

One obviously knows that the construction of efficient numerical approximations of path-dependent functionals F of X such as discounted price functionals

$$F_{r,p,X}(t,T) \;=\; \mathbb{E}\left[\exp\Big(-\int_t^T r(s)ds\Big)p((X(s))_{0\leq s\leq T})\Big|\mathcal{F}_t\right] \tag{2}$$

at exercise times $0 \leq t \leq T$ (T time of maturity) is important in the theory of dynamic asset pricing. Here $(r(t))_{t\geq 0} \geq 0$ is interpreted as an interest rate and p as a Borel-measurable functional on the risky asset price $(X(t))_{t\geq 0}$. The simplest and most cited example in finance is that of constant nonrandom interest rate r (or r satisfying SDEs such as (1)) and non-differentiable, but convex pricing functional

$$p((X(s))_{0\leq s\leq T}) = (X(T) - K)_+ \tag{3}$$

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where K is the striking price, T time of maturity and $(.)_+$ denotes the nonnegative part of inscribed expression. This occurs in the European call and put options. Others are given by look-back, Russian and Asian options.

Unfortunately, there are only very few models which allow to compute the price functionals F analytically. So one has to resort to numerical techniques to approximate F and p in general. Many authors have dealt with methods for numerical integration of solutions of Itô-type SDEs (1) and its functions $F(X(T))$ at fixed terminal time T. For example, see Allen [1], Artemiev and Averina [3], Bouleau and Lépingle [6], Gard [10], Kloeden, Platen and Schurz [14], Milstein [17], Schurz [25, 27, 31], Talay [33, 34, 35], Wagner and Platen [36]. Almost all of their methods are based on the classic Taylor expansion and its Runge-Kutta-type substitutions. However, most of those methods sometimes lack of rigorous statements on stability, positivity and convergence when complex nonlinearities, convexity or path-dependence in F are present.

The aim of this paper is to show how one can have a "minimal guarantee" of convergence and qualitative justification of numerical integration techniques which are needed to approximate functionals F such as given by (2) or similar ones under non-smooth assumptions or path-dependence. For this purpose, we shall exploit known and more easily verifiable facts on L^p-convergence rates. There are several good reasons why we prefer to use nonstandard implicit, strongly converging methods as originally introduced in [25, 18], studied in [23, 30] and continued by [19], [11], or even for quasilinear random PDE by [5]. Their good stability, boundary and positivity behavior is one of them. We shall justify these methods by studying how the rates of p-th mean convergence carry over to the rates of weak convergence along some functionals F despite non-smoothness or path-dependence. In particular, some new proof techniques come up by using integral representations of convex functions involving positive Radon measures. Another advantage is seen by the fact that we do not need to suppose very restrictive assumptions on the smoothness and boundedness of the coefficients of underlying SDEs as commonly met in the literature on stochastic numerics. This paper exhibits supplemental remarks to the results presented in Kanagawa and Ogawa [13] and Talay [33, 34]. Moreover, we do not focus too much on fairly known results which are supposed to be known to the readership. See Allen [1], Schurz [27], Talay [34] or appendix A for a quick overview on basic facts related to stochastic-numerical analysis.

The paper is organized as follows. Section 2 discusses how convergence rates of L^p-approximations carry over to rates of weak approximations while dealing with functionals involving convex functions. These estimates are only advantageous when not so much smoothness can be imposed on the functional F and its ingredients r, p and X (in contrast to standard requirements such as $p \in C^\infty$, p continuously differentiable or Lipschitz-continuous drift and diffusion coefficients of r and X). See [17, 34] for approximation rates of very smooth functions $F(X(T))$ (actually they only consider functions $F(X(T))$, not real functionals) or those F with non-degenerate infinitesimal generator of price process X. Section 3 proves a general theorem to control the total L^1-approximation error of path-dependent functionals such as F in (2). We also state a theorem on convergence of Hölder-continuous functionals. Eventually, we list numerous examples of functionals involving convex structures in Section 4. An appendix (Sections A.1 - A.3) resumes basic facts on numerical methods for Itô SDEs and its concepts convergence to increase the understanding of a more general audience. All in all, this paper presents just a supplemental discussion on some more complex issues related to numerical