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AN ALGORITHM-DRIVEN APPROACH TO ERROR ANALYSIS FOR MULTIDIMENSIONAL INTEGRATION

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Abstract. Most error analyses for numerical integration algorithms specify the space of integrands and then determine the convergence rate for a particular algorithm or the optimal algorithm. This article takes a different perspective of specifying the convergence rate and then finding the largest space of integrands for which the algorithm gives that desired rate. Both worst-case and randomized error analyses are provided.

Key Words. Digital nets, integration lattices, randomized, worst-case.

1. Introduction

Multi-dimensional integrals of the form

$$I(f) := \int_{\mathcal{X}_s} f(\mathbf{x})\, d\rho(\mathbf{x}), \qquad \mathcal{X}_s \subseteq \mathbb{R}^s, \tag{1}$$

arise in a number of applications. Here f is some known integrand and ρ is a given probability measure, i.e., $I(1) = 1$. For example, if $f(\mathbf{x})$ is the discounted payoff of an exotic option, and $\mathbf{x} = (x_1, \ldots, x_s)$ dictates the changes in the prices of the underlying assets that determine the payoff, then the fair price of that option is the average discounted payoff, $I(f)$, where $\mathcal{X}_s = \mathbb{R}^s$ and ρ is a multivariate normal distribution.

Error analysis of numerical integration rules typically yields error bounds and asymptotic rates of convergence for a specified Banach space of integrands. This article proposes a different approach to analyzing numerical integration rules, namely by specifying the convergence rate and the algorithm and then finding the largest space of integrands for which the algorithm gives that desired rate.

The integration rules considered here take the form of a simple average of integrand values:

$$Q_0(f) := 0, \qquad Q_n(f) := n^{-1} \sum_{i=0}^{n-1} f(\mathbf{x}_i) \quad \text{for } n > 0, \tag{2}$$

where $\{\mathbf{x}_i\}$ is the design or set of nodes where the integrand is evaluated. The nodes may be deterministic or random, but they are assumed to be independent of the integrand, making (2) a linear rule. Adaptive rules are not considered. The design is assumed to be an infinite sequence of which one uses the first n points. In

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practice one may want to consider Q_n for some increasing sequence of non-negative integers, $\mathcal{N} = \{0, n_0, n_1, \ldots\}$. A typical example is $n_m = 2^m$. Among the familiar rules of the form (2) that are considered here are simple Monte Carlo rules [4], rules based on low discrepancy sequences, such as integration lattices [11, 14, 20] or digital (t, m, s)-nets [14, 15]. Smolyak rules are similar, with the difference that one replaces n^{-1} with more general weights $a_{i,n}$, hence the approach proposed in this paper could also be applied to such rules.

Researchers have expended considerable effort to understand the strengths and weaknesses of various numerical integration rules. This is typically done by fixing a Banach space of integrands, $\mathcal{F}$, with a norm $\|\cdot\|_{\mathcal{F}}$, and computing the worst possible error of a particular rule for integrands of norm no greater than unity:

$$\operatorname{err}(f, Q_n) := I(f) - Q_n(f),$$
$$e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_n) := \sup_{\|f\|_{\mathcal{F}} \le 1} |\operatorname{err}(f, Q_n)|, \quad n \in \mathcal{N}. \tag{3}$$

The quantity $e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_n)$ is called the worst-case error of Q_n. Then one attempts to determine the asymptotic rate of convergence of this quantity, i.e., to show that

$$C_L(s) g(n) \le \inf_{\substack{n' \in \mathcal{N} \\ n' \le n}} e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_{n'}) \le C_U(s) g(n), \quad n = 1, 2, \ldots, \tag{4}$$

for some function $g(n)$ that tends to zero as $n \to \infty$. Typically $g(n)$ is a negative power of n or a negative power of n times some power of $\log n$. When (4) holds, one may say that $e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_n) \asymp g(n)$. If only an upper bound is known, then one may say that $e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_n) = \mathcal{O}(g(n))$. It is also of interest to know how the error depends on the dimension, s, i.e., whether $C_U(s)$ and $C_L(s)$ can be made independent of s, or polynomial in s. This corresponds to the problems of strong tractability or tractability, respectively, provided that $e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_0) = 1$ and $g(n)$ decays polynomially in n^{-1}.

Knowing that a numerical integration rule has a particular convergence rate is not the full story. One would also like to know the convergence rate of the best possible rule. The worst-case difficulty of an integration problem can be defined as the error of the best possible rule:

$$e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, n) := \inf_{Q_n} e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_n), \quad n \in \mathcal{N}. \tag{5}$$

If $e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, Q_n) \asymp e^{\mathrm{wo}}(\|\cdot\|_{\mathcal{F}}, n)$, then the rule Q_n is optimal. In other words, an optimal integration rule has the same convergence rate as the best rule, but their errors may differ by a constant factor (which again may depend on s).

When the numerical integration rule used is a randomized one, then it makes sense to compute the randomized error. Let $\mathcal{Q}_n$ denote the sample space of random rules Q_n, where now n denotes the average number of function evaluations used. Let μ be a probability measure on this sample space, and let $\operatorname{rms}_{\mathcal{Q}_n}$ denote the root mean square using this measure μ. The randomized error for a given $\mathcal{Q}_n$ and μ is defined as

$$\operatorname{rmse}(f, \mathcal{Q}_n, \mu) := \operatorname*{rms}_{\mathcal{Q}_n} |\operatorname{err}(f, Q_n)|,$$
$$e^{\mathrm{ra}}(\|\cdot\|_{\mathcal{F}}, \mathcal{Q}_n, \mu) := \sup_{\|f\|_{\mathcal{F}} \le 1} \operatorname{rmse}(f, \mathcal{Q}_n, \mu), \quad n \in \mathcal{N}. \tag{6}$$

Although the norm-based approach described above is quite useful, it has a certain drawback that this article attempts to address, namely, the space of integrands $\mathcal{F}$ is fixed in advance. Once the space of integrands and the accompanying norm