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SOBOLEV GRADIENT TYPE PRECONDITIONING FOR THE SAINT-VENANT MODEL OF ELASTO-PLASTIC TORSION

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Abstract. In this paper a suitable Laplacian preconditioner is proposed for the numerical solution of the nonlinear elasto-plastic torsion problem. The aim is to determine the tangential stress in cross-sections under a given torsion, for which the physical model is based on the Saint Venant model of torsion and the single curve hypothesis for the connection of strain and stress. The proposed iterative solution of the arising nonlinear elliptic problem is achieved by combining the advantages of Laplacian preconditioners with the qualitatively favourable aspects of the strong formulation. Error estimate is given for the convergence of the method. Finally, a numerical example is given.

Key Words. Elasto-plastic torsion; nonlinear elliptic problem; iterative solution; Laplacian preconditioner.

1. Introduction

The investigation of the elasto-plastic torsion of a hardening rod has a great practical importance in mechanics and its theoretical background has been widely analysed (see, e.g., [11, 12]). The mathematical formulation of this problem leads to nonlinear differential equations. The most frequently used numerical methods that arise in this context are the finite difference and finite element methods [20, 23]. The solution of the obtained system of algebraic equations is generally found by some iterative method. The crucial point in the solution of these systems is most often preconditioning. Namely, since the condition number of the Jacobians of these systems tends to infinity when discretization is refined, therefore a suitable nonlinear preconditioning technique has to be used to achieve a convenient condition number [2].

In this paper the behaviour of the tangential stress is studied under the elasto-plastic torsion of a hardening rod based on the following model [12]: the cross-sections experience rigid rotation in their planes and are distorted in the direction of the z-axis (this is the Saint Venant model), further, the intensity of the stress is a strictly increasing function of that of the strain under the hardening condition. The arising mathematical model is a quasilinear elliptic boundary value problem of divergence form, in which nonlinearity comes from the stress-strain function.

As mentioned above, the main point in the numerical solution of the arising elliptic problem is preconditioning. A general efficient way to provide a suitable

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preconditioner is the Sobolev gradient approach, developed by Neuberger for least-square methods [21, 22], which relies on using the Sobolev inner product. A strongly related kind of preconditioning is using the discrete Laplacian as preconditioner (see e.g. [7, 25]). These preconditioning methods benefit by the fast solvers available for the Laplacian, also involving general domains via the fictious domain approach (see also [25]). The Sobolev gradient technique points to the infinite-dimensional generalizations of iterative methods, which go back to Kantorovich [13] and have undergone extensive development. The authors' investigations include the gradient method for non-differentiable operators in a Hilbert space [14], and we underline that the Sobolev space background helps us in constructing effective natural preconditioners [3, 10, 21].

The Sobolev gradient approach yields a gradient (steepest descent) iteration in Sobolev space which reduces the solution of the nonlinear equation to the sequence of auxiliary linear Poisson problems. The numerical solution of these auxiliary linear problems by a suitable finite element method yields the gradient–finite element method (GFEM) introduced by the authors in [9]. This method combines the above mentioned advantages of Laplacian preconditioners with the qualitatively favourable aspects of the strong formulation. The GFEM is proposed in the present paper for the numerical solution of the elasto-plastic torsion problem. The main advantages of the GFEM are an easy algorithmization and preserving the ellipticity bounds of the differential operator in the ratio of linear convergence. The latter provides a priori mesh independent estimates for the FEM realization and is due to the above-mentioned Sobolev space preconditioning background.

Besides the GFEM, we will sketch some other applications of Laplacian preconditioners. In the comparison to other numerical methods it is important to refer to Newton's method, widespread for its fast convergence. The problem of only local convergence and the extra work of compiling the Jacobians may justify the choice of a theoretically slower method, cf. e.g. [3]. In the GFEM the auxiliary linear problems are of fixed (Poisson) type, hence the matrices need not be updated in each step. Further, as we will see, in our problem the rate of linear convergence is suitably small. We note that for problems where the Laplacian preconditioner cannot yield a favourable ratio of convergence, one can still use the Sobolev space setting to construct preconditioned Newton iterations [4, 16, 25]. Some further remarks on the comparison of the GFEM, the nonlinear CGM and Newton's method will be given in Subsection 4.2.

The paper is organized as follows. Section 2 describes the physical model based on [12]. In Section 3 mathematical background is given. Section 4 is devoted to the construction and error estimate of the GFEM and some related applications of Laplacian preconditioners. Finally, in Section 5 numerical realization is developed for computing the tangential stress in a copper bar when crack occurs.

2. The physical model of 2D elasto-plastic torsion

The mathematical model of plastic state under plane deformation conditions was first given by Saint-Venant, and was later extended by von Mises to 3D, having a simple physical interpretation and structure.

In the hardening state the model of elasto-plastic torsion is given below following the presentation of Kachanov [12]. This model is based on the observation that the equations of deformation theory may be used for plastic deformations which develop in some definite direction. Since the tangential stress vectors act in parallel cross-sections, the model reduces the 3D problem to 2D.