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THE REGULARIZATION METHOD FOR A DEGENERATE PARABOLIC VARIATIONAL INEQUALITY ARISING FROM AMERICAN OPTION VALUATION

GUANGHUI WANG[1] AND XIAOZHONG YANG[2]

Abstract. In this paper, we present a regularization method to a degenerate variational inequality of parabolic type arising from American option pricing. Main difficulty in actually analyzing this kind of problem is caused by the presence of a non-smoothing initial value function in the formulation of the problem. We first use a smoothing technique with small parameter $\varepsilon > 0$ to non-smoothing initial value function; and then we derive the error estimates for regularized continuous problem and regularized discrete problem, respectively. Numerical tests are given to confirm our theoretical results.

Key Words. regularization method, variational inequality, American Option valuation, finite element and error estimates.

1. Introduction

Option trading forms part of our financial markets. A traded option gives to its owner the right to buy (*call option*) or to sell (*put option*) a fixed quantity of assets of a specified stock at a fixed price (*exercise* or *strike price*). There are two major types of traded options. One is the *American option* that can be exercised at any time prior to its *expiry date*, and the other option, which can only be exercised on the expiry date, is called the *European option*. It was shown by Black and Scholes (cf. [3]) that the value of an European option is governed by a second order parabolic differential equation with respect to time and the underlying stock price. This is now referred to as the Black-Scholes equation. The value of an American option is governed by a more complex mathematical model due to the flexibility on exercise date. It can be shown that American option pricing is determined by a linear complementarity problem involving the Black-Scholes differential operator and a constraint on the value of the option (cf., for example, [20, 19]). This complementarity problem can also be formulated as a variational inequality (cf. [19]). The Black-Scholes equation is a degenerate partial differential as its coefficients of the first and second order spatial derivatives vanish as the underlying stock price approaches zero. A popular method of removing this difficulty is to introduce a new variable and transform the Black-Scholes equation into a heat equation defined on the whole real number set. This technique is used in many existing papers such as [1, 10, 20]. In this case, the degeneracy point is transformed to $-\infty$. However, when solve the resulting heat equation numerically,

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the infinite horizon is truncated to a finite region. Recently, a fitted finite volume method is proposed in [18] to handle the degeneracy, based on the idea in [13, 14]. This technique can also be used for solving the American option problem if it is used along with a power penalty method (cf., for example, [19]).

In this paper we shall discuss the regularization method [12] for solving the parabolic variational inequality with a degenerate partial differential operator governing American option valuation. To our best knowledge, there are relatively few papers in which numerical methods are studied for parabolic variational inequalities (cf, for example, [1, 7, 17] and references therein), let alone parabolic inequalities with degenerate partial differential operators (cf. [8]). The main difficulty is that solutions to parabolic variational inequalities normally less smooth than those of elliptic problems even all the data are smooth. Johnson [7] and Vuik [17] studied the finite element approximations of a variational inequality of parabolic type under some regularity assumptions on the exact solution. To bypass the difficulty, we shall construct a regularization method for the variational inequality involving the Black-Scholes operator, and derive the error bound in the weighted Sobolev space for the method.

The remainder of this paper is organized as follows. In the next section we will state the strong problem governing American put option pricing and some preliminaries. In Section3, we shall rewrite the problem as a more mathematical form, i.e., a variational inequality, and discuss the solvability of the resulting problem. In section4, we present the regularity problem of problem3.1, and prove its error bound with ε. We will present some numerical results to confirm the theoretical findings in Section5.

2. Preliminaries

Let V denote the value of an American put option with strike price K and expiry date T, and let x be the price of the underlying asset of the option. It is known (cf., for example, [20]) that V satisfies the following strong form of linear complementarity problem

$$LV(x,t) \geq 0, \tag{2.1}$$

$$V(x,t) - V^*(x) \geq 0, \tag{2.2}$$

$$LV(x,t)\cdot(V(x,t) - V^*(x)) = 0, \tag{2.3}$$

a.e. in $\Omega := I \times J$, where L is the Black-Scholes operator defined by

$$LV := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 V}{\partial x^2} - r(t)x\frac{\partial V}{\partial x} + r(t)V, \tag{2.4}$$

$I = (0, X) \subset R$ and $J = (0, T)$ with positive constants X and T, $\sigma(t)$ denotes the volatility of the asset, $r(t)$ the interest rate, and V^* is the final (payoff) condition defined by

$$V(x,T) = V^*(x) = \max\{K - x, 0\}. \tag{2.5}$$

For clarity, we only consider American put options in this paper. Naturally, the theory developed applies to American call options and other complementarity problems of the form (2.1)–(2.3) arising in finance as well.

Some standard notation is to be used in the paper. For an open set $S \in R$ and $1 \leq p \leq \infty$, we let $L^p(S) = \{v : (\int_S |v(x)|^p dx)^{1/p} < \infty\}$ denote the space of all p-power integrable functions on S. The inner product and the norm on $L^2(S)$ are denoted respectively by $(\cdot,\cdot)_S$ and $\|\cdot\|_0$. We use $\|\cdot\|_{L^p(S)}$ to denote the norm on $L^p(S)$. For $m = 1, 2, ...,$ we let $H^{m,p}(S)$ denote the usual Sobolev space with