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SUPERCONVERGENT TECHNIQUES IN MULTI-SCALE METHODS

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Abstract. It is well known that many problems of practical importance in science and engineering have multiple-scale solutions. Moreover, the calculations of numerical methods for these problems is very intensive, even if using some multi-scale proceedures. It is therefore important to seek efficient calculation methods. In this paper, superconvergent techniques are used in existing multi-scale methods to improve the calculation efficiency. Furthermore, based on comprehensive analysis, the order of the error estimates between the numerical approximation and the exact solution is verified to be improved.

Key Words. Elliptic equations, superconvergent technique, periodic microstructure, multi-scale methods, asymptotic expansion, homogenization.

1. Introduction

Multi-scale methods have been investigated for a long time in the mathematics and engineering literature. For example of these papers, we refer to [4], [9] and [11]. Early papers concentrated on multi-scale methods that are mainly based on the theory of asymptotic expansion and homogenization. Later, various different but related multi-scale methods were proposed, including the multigrid numerical homogenization method ([33], [34], [46], [47]), the multiscale finite element method (MsFEM) ([37], [38], [31]), the heterogeneous multiscale method (HMM) ([25], [26], [27], [28]), the finite element method based on the *Residual-Free Bubble* method ([12], [32], [35], [39]), the wavelet homogenization method ([22]) and so on. Each of these methods has advantages in some special cases. As is well known, the multi-grid method as a classical multi-scale technique achieves optimal efficiency by relaxing the errors at different scales on different grids. It can give an accurate approximation to the detailed solution of fine scale problems. HMM is a specific strategy to compute the macro-scale behavior of the system with a standard macro-scale scheme in which the missing micro-scale data can be evaluated concurrently by using the micro-scale model. It can deal with many multi-scale problems efficiently even for problems whose period is unknown. MsFEM can obtain the large scale solutions accurately and efficiently without resolving the small scale details. The main idea is to construct in each element finite base functions which can capture the small scale information. Such small-scale information is then brought to the large scales through the coupling of the global stiffness matrix.

Although the methods can deal efficiently with some practical problems, the computation cost may still be very large. For example, in order to simulate elliptic problems with non-uniformly oscillating coefficients by HMM, at least one unit cell

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in each element will be calculated to obtain the homogenized equation and obtain the information of the microstructure. This results in intensive calculations if the number of elements is large. In some cases where the domain and the solution are smooth enough, it is important to find a more efficient method or technique to reduce the calculations. It is known that, in [13], a fast post-processing algorithm which is based on asymptotic expansion used to analyze a multiscale method. But in [13], the authors just analyzed elliptic problems with uniformly highly oscillatory coefficients. In practice, there are many multiscale problems with non-uniformly oscillating coefficients, and by using the post processing technique directly, it is impossible to improve the order of the error estimate on the whole domain when one just uses linear interpolation for the unit cells that have been simulated. For instance, under the conditions above, the error estimate of the HMM for the H^1-broken norm is just $O(H)$. If we use a high order interpolation technique, then the number of unit cells needed in the calculations will increase greatly if the HMM method is employed. So, it is very important to reduce the number of unit cells needed in the calculations. In this paper, we show that it is not necessary to choose at least one unit cell in each element in which to calculate. We simulate unit cells on a new mesh, which is different from the partition of the whole domain. The size of the former is much bigger than that of the latter. This idea is different from that used in HMM and some other multiscale methods. By using high order interpolation techniques for the solved unit cells, we then successfully reduce the cost on unit cells. Moreover, we can use a superconvergent technique to deal with the numerical solution of the homogenized equation in order to improve its accuracy. Based on these ideas, some improved error estimates are given. In this paper, we just investigate the superconvergent techniques in the homogenized equations presented in [9] and [27]. In fact, superconvergent techniques can also be efficiently extended to some other multiscale methods. In addition, we just discuss elliptic problems. For parabolic multiscale problems with suitable conditions, the superconvergent technique is also valid.

In the past forty years, superconvergence finite element methods has been an active research field. Early papers concentrated on superconvergence at isolated points (see [23] *et al*). Later, various type of superconvergent techniques were established, either in the strict sense or in an approximate way (see [7], [8], [52], [53], [58], [59], [60], [40], [44] *et al*). In this paper, we merely give a framework to demonstrate that the superconvergent technique is suited to multi-scale methods and can efficiently improve the accuracy. Thus, we only employ certain postprocessing techniques proposed in [40] and [44] to improve the existing approximation accuracy. However some other superconvergent techniques, such as the *Zienkiewicz-Zhu superconvergent Patch Recovery* (ZZ-SPR), can also be used to improve the order of error estimates of multi-scale methods.

The outline of this paper is as follows. In the next section, we introduce the model problem and provide its two similar homogenized equations. Moreover, the error estimate between the exact solution of the original problem and the asymptotic expansion of order one is presented, and the estimates

$$\|u^\epsilon - u_1^\epsilon\|_{1,D} \le C\sqrt{\epsilon}\|U_0\|_{3,\infty,D}$$

$$\|u^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le (Ch^k\|u_0\|_{1,D} + \sqrt{\epsilon}\|u_0\|_{3,\infty,D}),$$

are obtained.

Based on this result, we present the principal results of this paper in Section 3. The error estimate, between the exact solution and the numerical solution of the