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SOLVING SINGULARLY PERTURBED REACTION DIFFUSION PROBLEMS USING WAVELET OPTIMIZED FINITE DIFFERENCE AND CUBIC SPLINE ADAPTIVE WAVELET SCHEME

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Abstract. In this paper singularly perturbed reaction diffusion equations of elliptic and parabolic types have been discussed using wavelet optimized finite difference (WOFD) method based on an interpolating wavelet transform using cubic spline on dyadic points as discussed in [1]. Adaptive feature is performed automatically by thresholding the wavelet coefficients. WOFD [2] works by using adaptive wavelet to generate an irregular grid which is then exploited for the finite difference method. Numerical examples are presented for elliptic and parabolic problems and comparisons have been made using cubic spline and WOFD. The proposed adaptive method is very effective for studying singular perturbation problems in term of adaptive grid generation and CPU time.

Key Words. Singularly perturbed reaction diffusion problems, WOFD, Splines wavelets, Multiresolution analysis, Fast discrete wavelet transform, Lagrangian finite difference.

1. Introduction

In this paper cubic spline and WOFD have been applied for solving singularly perturbed reaction diffusion problems of elliptic and parabolic types. Problems in which a small parameter is multiplied to the highest derivative arise in various fields of science and engineering, for instance fluid mechanics, elasticity, hydrodynamics, etc. The main concern with such problems is the rapid growth or decay of the solution in one or more narrow "layer region(s)". The specific problems under consideration in this paper is called dissipative because the rapid varying component of the solution decays exponentially (dissipates) away from a localized breakdown in the layer region(s) as $\epsilon \to 0$.

Here the author is trying to use cubic spline adaptive wavelet features to solve singularly perturbed reaction diffusion problems belonging to sobolev space $\mathcal{H}_0^2(\mathcal{I})$. Cai and Wang's [1] wavelets have been chosen because of their interpolating property. They were considered by the authors as "a semi-orthogonal cubic spline wavelet basis of homogenous sobolev space $\mathcal{H}_0^2(\mathcal{I})$".

Two semilinear singularly perturbed reaction diffusion problem of elliptic and parabolic types have been discussed. The first one is the elliptic problem defined as

$$-\epsilon u_\epsilon''(x) = f(x, u_\epsilon) \text{ where } 0 \leq x \leq 1, \tag{1a}$$

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with boundary conditions

$$u_\epsilon(0) = u_a,\ \ u_\epsilon(1) = u_b,\ \ \frac{\partial f}{\partial u_\epsilon} \geq c^2,\ c = \text{const.} > 0, \tag{1b}$$

where ϵ is a small parameter and $f(x, u)$ is sufficiently smooth. For $\epsilon \ll 1$, the solution has boundary layers at $x = 0$ and $x = 1$ [3].

The second problem is the one dimensional parabolic problem

$$-\epsilon\, u_{xx}(x, t) + a(x, t)u(x, t) + b(x, t)u_t(x, t) = f(x, t, u); \tag{2a}$$

where $(x, t) \in Q = (0, 1) \times (0, T]$ and

$$u(x, 0) = s(x) \text{ on } S_x = (x, 0) :\ 0 \leq x \leq 1, \tag{2b}$$

$$u(0, t) = q_0(t) \text{ on } S_0 = (0, t) : 0 < t \leq T, \tag{2c}$$

$$u(1, t) = q_1(t) \text{ on } S_1 = (1, t) : 0 < t \leq T. \tag{2d}$$

Here ϵ is a parameter satisfying $0 < \epsilon \ll 1$. We assume that $a(x, t) \geq a_0 > 0$ and $b(x, t) \geq b_0 > 0$ on $\overline{Q}$, where a_0, b_0 are some constants and $\overline{Q} = [0, 1] \times [0, T]$ denotes the closure of Q. The solution u has in general boundary layers of parabolic type along the sides $x = 0$ and $x = 1$ of $\overline{Q}$ [4].

Singular perturbation problems in consideration have shocks as boundary layers. For such kinds of problems the solution can be smooth in most of the solution domain with small area where the solution changes quickly. When solving such problems numerically, one would like to adjust the discretization to the solution. In term of mesh generation, we want to have many points in area where the solution have strong variations and few points in area where the solution is smooth. Elliptic problem earlier has been discussed [5] using cubic B-spline with Shishkin mesh. One should have the pre-knowledge about the locations of the boundary layers for working with Shishkin meshes, therefore, also motivates us for adaptive methods for effective grid generation.

Wavelets have been making an appearance in many pure and applied area of science and engineering [2]. Wavelets detect information at different scales and at different locations throughout a computational domain. Wavelets can provide a basis set in which the basis functions are constructed by dilating and translating a fixed function known as the mother wavelet. The mother wavelet can be seen as a high pass filter in the frequency domain. One of the key strength of the wavelet methods is data compression. An efficient basis is one in which a given set of data can be represented with as few basis elements as possible. Given a wavelet representation of a function

$$\sum_k c_{j,k}\varphi_{j,k}(x) + \sum_{j,k} d_{j,k}\psi_{j,k}(x),$$

where $\varphi_{j,k}(x)$ are scaling functions and $\psi_{j,k}(x)$ are wavelets, the scaling function coefficients $c_{j,k}$, essentially encode the smooth part of the function, while the wavelet coefficient $d_{j,k}$ contains information of the function behavior on successive finer scales. The most common way of compressing such a representation is thresholding. We delete all wavelet coefficients of magnitude less than some threshold, say τ. If the total no. of coefficients in the original representation was N, we have N_a significant coefficients left after the thresholding. Note that by thresholding a wavelet representation we have a way to find an adaptive feature and we can also use this representation to compute function values at any point.