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A UNIFORMLY OPTIMAL-ORDER ERROR ESTIMATE OF AN ELLAM SCHEME FOR UNSTEADY-STATE ADVECTION-DIFFUSION EQUATIONS

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Abstract. We prove an optimal-order error estimate in a weighted energy norm for the Eulerian-Lagrangian localized adjoint method (ELLAM) for unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions. It is well known that these problems admit dynamic fronts with interior and boundary layers. The estimate holds uniformly with respect to the vanishing diffusion coefficient.

Key Words. characteristic methods, Eulerian-Lagrangian methods, interpolation of spaces, uniform error estimates

1. Introduction

We consider unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions, which arise in mathematical modeling of petroleum reservoir simulation, environmental modeling, and other applications [1, 7]. It is well known that these problems admit solutions with dynamic fronts and complex structures including interior and boundary layers, and present serious mathematical and numerical difficulties. Classical finite difference or finite element methods tend to generate numerical solutions with nonphysical oscillations, while upwind methods often produce excessive numerical diffusion that smears out fronts and generates spurious grid orientation effects [7].

Eulerian-Lagrangian methods combine the advection and capacity terms in the governing equations to carry out the temporal discretization in a Lagrangian coordinate, and discretize the diffusion term on a fixed mesh in an Eulerian manner [4, 5, 11]. These methods symmetrize the governing equation and stabilize their numerical approximations. They generate accurate numerical solutions and significantly reduce the numerical diffusion and grid-orientation effect present in upwind methods, even if large time steps and coarse spatial meshes are used. Eulerian-Lagrangian methods were shown to be very competitive in terms of accuracy and efficiency [4, 12]. Mathematically, *A priori* optimal-order error estimates were derived for the modified method of characteristics (MMOC) [5] and the modified method of characteristics with adjusted advection [4] for unsteady-state advection-diffusion equations with periodic or noflow boundary conditions and the Eulerian-Lagrangian localized adjoint method (ELLAM) for unsteady-state advection-diffusion equations with general boundary conditions [13, 10]. However, the general constant in this type of estimates may depend inversely on the vanishing diffusion parameter. Consequently, these estimates could blow up as the diffusion coefficient tends to zero. To our best knowledge, there is no *a priori* optimal-order error estimate in a

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weighted energy norm for an Eulerian-Lagrangian method with uniform partition for unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions, which holds uniformly with respect to the vanishing diffusion parameter.

In contrast to the steady-state analogue where a uniform L^∞ error estimate was derived for numerical methods with a Shishkin mesh [9], unsteady-state advection-diffusion equations admit dynamic interior and boundary layers and complicated structures. These boundary and interior layers are dynamic and do not always coincide with the spatial mesh. Consequently, a uniform error estimate in the L^∞-norm is generally impossible, since the true solution could exhibit shock discontinuity in the limiting case of the diffusion parameter vanishes. This is why L^∞ norm is not used in the numerical analysis for hyperbolic conservation laws [8]. The goal of the present paper is to derive an optimal-order error estimate in a weighted energy norm for the ELLAM scheme for unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions. Thus, these results theoretically justify the numerical advantages of Eulerian-Lagrangian methods, which were observed numerically [11, 12, 13].

This paper is organized as follows. Sections 2 and 3 recall preliminary results on Sobolev and interpolation results and revisit the ELLAM scheme, respectively. In this section 4, we prove an ε-uniform optimal-order error estimate in a weighted-energy norm for the ELLAM scheme for unsteady-state advection-diffusion equations with an inflow total flux and an outflow diffusive flux boundary conditions, which admit both interior and boundary layers. In section 5, we prove auxiliary estimates that were used in the proof in section 4.

2. Model Problem and Preliminaries

We consider the unsteady-state advection-diffusion equation in one space dimension with a representative combination of an inflow total flux boundary condition and an outflow diffusive flux boundary condition. The analysis in this paper applies to any combinations of boundary conditions. For the sake of exposition, we restrict ourselves to this representative combination of boundary conditions, which is well known to present mathematical and numerical difficulties in the theoretical analysis of Eulerian-Lagrangian methods [13]

$$
\begin{array}{rclcrcl}
u_t + (V(x,t)u - \varepsilon D(x,t)u_x)_x & = & f(x,t), & & (x,t) & \in & (a,b)\times(0,T) \\
Vu(a,t) - \varepsilon D u_x(a,t) & = & g(t), & & t & \in & (0,T] \\
-\varepsilon D u_x(b,t) & = & h(t), & & t & \in & (0,T] \\
u(x,0) & = & u_o(x), & & x & \in & [a,b].
\end{array}
\tag{1}
$$

Here $V(x,t)$ is a velocity field, $f(x,t)$ accounts for external sources and sinks, $g(t)$ and $h(t)$ are the prescribed inflow and outflow boundary data, respectively, $u_o(x)$ is the prescribed initial data, and $u(x,t)$ is the ε-dependent unknown function. $D(x,t)$ is a diffusion coefficient with $0 < D_{min} \le D(x,t) \le D_{max} < +\infty$ for any $(x,t) \in [a,b]\times[0,T]$ and $0 < \varepsilon << 1$ is a parameter that scales the diffusion and characterizes the advection-dominance of Eq. (1).

Let $W^k_p(a,b)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in (a,b). Let $H^k(a,b) := W^k_2(a,b)$. For any Banach space X,