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EXACT DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS

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Abstract. The Cauchy problem for the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\,(x,t)\,\frac{\partial u}{\partial x}\right) + f(u,x,t),\ x \in R,\ t > 0,$$
$$u(x,0) = u_0(x),\ x \in R,$$

is considered. Under conditions $u(x,t) = X(x)T_1(t) + T_2(t)$, $\frac{\partial u}{\partial x} \neq 0$, $k(x,t) = k_1(x)k_2(t)$, $f(u,x,t) = f_1(x,t)f_2(u)$, it is shown that the above problem is equivalent to a system of two first-order ordinary differential equations for which exact difference schemes with special Steklov averaging and difference schemes with any order of approximation are constructed on the moving mesh. On the basis of this approach, the exact difference schemes are constructed also for boundary-value problems and multi-dimensional problems. Presented numerical experiments confirm the theoretical results investigated in the paper.

Key Words. exact difference scheme, difference scheme with an arbitrary order of accuracy, parabolic equation, system of ordinary differential equations.

1. Introduction

Various schemes have been constructed to approximate initial- and boundary-value problems for parabolic equations [17]. One of the main questions in investigating difference schemes is the approximation order, which is desired to be as high as possible.

In the last few years, the exact difference schemes for some partial differential equations have been constructed [3], [5], [6]. It is worth here to mention the papers by R.E. Mickens [7] - [12], in which certain rules for construction of the nonstandard finite difference schemes are given and several such schemes were introduced, for example for the Burgers partial differential equation having no diffusion and a nonlinear logistic reaction term [11]. S. Rucker [16] applied techniques initiated by R. E. Mickens to obtain exact difference scheme for an advection – reaction equation. In the paper [5], under natural conditions, the authors proved existence of a two-point exact difference scheme for systems of first-order boundary value problems. Difference schemes of high order of approximation were also constructed in [15], [19].

The authors earlier established that for problems for parabolic equations with solutions of the separated variables $u(x,t) = X(x)T_1(t) + T_2(t)$ the exact difference scheme may be constructed. The main feature of this paper is to apply the method introduced in [6] for a wider classes of problems. The attention is mainly devoted

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to constructing a difference scheme of arbitrary order of approximation in the case when the integral in special Steklov averaging cannot be evaluated exactly, as well as developing the exact difference schemes in multi-dimensional case by using the presented approach.

Consider the Cauchy problem for the one-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\,(x,t)\,\frac{\partial u}{\partial x}\right) + f(u,x,t),\ x \in R,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x),\ x \in R. \tag{1.2}$$

Under conditions $u(x,t) = X(x)T_1(t) + T_2(t)$, $\frac{\partial u}{\partial x} \neq 0$, $k(x,t) = k_1(x)k_2(t)$, $f(u,x,t) = f_1(x,t)f_2(u)$, we show that problem (1.1) - (1.2) is equivalent to the following system of two ordinary differential equations [6]:

$$\frac{dx}{dt} = c_1(x)k_2(t), \tag{1.3}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=c_1(x)k_2(t)} = f_1(x(t),t)f_2(u),\ u(x(0),0) = u_0(x(0)), \tag{1.4}$$

where $c_1(x) = -\frac{\left(k_1(x)u_0'(x)\right)'}{u_0'(x)}$. ¿From (1.3) we find the curve $x = x(t)$, along which we get from (1.4) the solution $u(x,t) = u(x(t),t)$ of problem (1.1) - (1.2). Here $x(0) = x^0 \in R$ is the initial state of the curve $x = x(t)$. Special Steklov averaging [6], [17]

$$c(x(t)) \approx \left[\frac{1}{x^{n+1} - x^n}\int\limits_{x^n}^{x^{n+1}} \frac{dx}{c(x)}\right]^{-1},\ t_n \le t \le t_{n+1},\ x^n = x(t_n),\ t_n = n\tau$$

is used to construct exact difference schemes only on the moving mesh. On the basis of this approach, the exact difference schemes are constructed also for boundary-value problems and for multi-dimensional problems. A difference scheme of arbitrary order of approximation is proposed in the case when the integral in the Steklov averaging cannot be evaluated exactly.

2. Exact difference schemes: the Cauchy problem for parabolic equations

In this Section, using the special Steklov averaging the exact difference scheme for the Cauchy problem for parabolic equations is constructed.

Let us consider in the domain $Q_T = R \times [0,\infty)$ the Cauchy problem for the one-dimensional parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + f(x,t,u),\ x \in R,\ t > 0, \tag{2.5}$$

$$u(x,0) = u_0(x),\ x \in R. \tag{2.6}$$

Assume that the problem (2.5) - (2.6) has an unique solution $u(x,t) \in C_1^2(Q_T)$, $u(x,t) = X(x)T_1(t) + T_2(t)$, $\frac{\partial u}{\partial x} \neq 0$ and that the input data has the following form $k(x,t) = k_1(x)k_2(t)$, $f(x,t,u) = f_1(x,t)f_2(u)$. The coefficient k is bounded from above and below, i.e. $0 < k_1 \le k(x,t) \le k_2$, for $(x,t) \in R \times (0,\infty)$, where $k_1,\ k_2 = const$, and $k(x,t) \in C_1^1(Q_T)$.