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LOW ORDER CROUZEIX-RAVIART TYPE NONCONFORMING FINITE ELEMENT METHODS FOR APPROXIMATING MAXWELL'S EQUATIONS

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Abstract. The aim of this paper is to study the convergence analysis of three low order Crouzeix-Raviart type nonconforming rectangular finite elements to Maxwell's equations, on a mixed finite element scheme and a finite element scheme, respectively. The error estimates are obtained for one of above elements with regular meshes and the other two under anisotropic meshes, which are as same as those in the previous literature for conforming elements under regular meshes.

Key Words. Maxwell's equations, low order nonconforming finite elements, error estimates, anisotropic meshes.

1. Introduction

It is well-known that Maxwell's equations are very important equations in the electric-magnetic fields and are usually solved with finite element methods(see [1-8]). P.Monk [2-4] described a mixed finite element scheme and a finite element scheme, respectively, and provided convergence and superconvergence analysis for smooth solutions for three-dimensional Maxwell's equations. Lin and Yan [5] improved Monk's results by means of the technique of integral identity. The similar results were proved for two-dimensional Maxwell's equations by Lin [6,8] and Brandts [7].

However, there are still some defects in the work mentioned above. On the one hand, all of previous analysis only concentrated on conforming finite elements, for examples, ECHL element, MECHL element, N*édé*lec's element [1] and so on. Whether those results hold for nonconforming ones or not is still an open problem. On the other hand, to our best knowledge, almost all the convergence analysis in the literature on this aspect are based on the classical regularity assumption or quasi-uniform assumption on the meshes [9], i.e., $\frac{h_K}{\rho_K} \le C$ or $\frac{h}{h_{min}} \le C$, $\forall K \in T_h$, where T_h is a family of meshes of Ω, h_K and ρ_K are the diameter of K and the biggest circle contained in the element K, respectively, $h = \max\limits_{K\in T_h} h_K, h_{min} = \min\limits_{K\in T_h} h_K$ and C is a positive constant which is independent of h_K and the function considered. However, in many cases, the above regular assumptions on meshes are great deficient in applications of finite element methods. For example, the solutions of some elliptic problems may have anisotropic behavior in parts of the defined domain. This means that the solution only varies significantly in certain directions. It is an

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obvious idea to reflect this anisotropy in the discretion by using anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution and a coarser mesh size in the perpendicular direction. Besides, some problems may be defined in narrow domain, for example, in modeling a gap between rotor and stator in an electrical machine, if we employ the regular partition of the domain, the cost of calculation will be very high. Therefore, to employ anisotropic meshes with fewer degrees of freedom is a better choice to overcome these difficulties. However, anisotropic elements K are characterized by $\frac{h_K}{\rho_K} \to \infty$, where the limit can be considered as $h \to \infty$. In this case, the Bramble-Hilbert Lemma can not be used directly in the estimate of the interpolation error, and the consistency error estimate. The key of the nonconforming finite element analysis, will become very difficult to be dealt with, because there will appear a factor $\frac{|F|}{|K|}$ which tends to ∞ when the estimate is made on the longer sides F of the element K, which means that the traditional finite element analysis techniques are no longer valid. Zenisek [10,11] and Apel [12,13] published a series of papers concentrating on the interpolation error estimates of some Lagrange type conforming elements, and [13] represented an anisotropic interpolation theorem, but it is very difficult to be verified for some elements. Chen and Shi [14] generalized Apel's results and studied many problems, including anisotropic nonconforming elements, and obtained a lot of valuable results [14-19]. Although anisotropic finite element methods have such obvious advantages over conventional ones, it seems that there are few studies focusing on Maxwell's equations of the finite element formulations, especially the nonconforming ones.

In this paper, we will apply three Crouzeix-Raviart type nonconforming finite elements (one is the so-called five-node nonconforming element[15,20], another is similar to the so-called P_1 nonconforming finite element discussed in [21] and the last one is a new element constructed in this paper) to Maxwell's equations on a mixed finite element scheme and a finite element scheme, respectively. The plan of this paper as follows: in section 2, we will give the constructions of the three Crouzeix-Raviart type nonconforming finite elements, analyze the mixed finite element scheme and the finite element scheme for the time-dependent Maxwell's system in two dimensions and prove some important Lemmas. In section 3, the so-called five-node nonconforming element is applied to Maxwell's equations on the finite element scheme, meanwhile, the other two elements are applied to approximating Maxwell's equations on the mixed finite element scheme and the finite element scheme, respectively. Based on some novel approaches and elements' properties, the convergence analysis and error estimates are obtained for two elements under anisotropic meshes and the other one with regular meshes, respectively.

2. Constructions of nonconforming finite element schemes

Let $\hat{K} = [-1,1]\times[-1,1]$ be the reference element on $\xi-\eta$ plane, the four vertices of $\hat{K}$ are $\hat{d}_1 = (-1,-1)$, $\hat{d}_2 = (1,-1)$, $\hat{d}_3 = (1,1)$ and $\hat{d}_4 = (-1,1)$, the four edges of $\hat{K}$ are $\hat{l}_1 = \overline{\hat{d}_1\hat{d}_2}$, $\hat{l}_2 = \overline{\hat{d}_2\hat{d}_3}$, $\hat{l}_3 = \overline{\hat{d}_3\hat{d}_4}$ and $\hat{l}_4 = \overline{\hat{d}_4\hat{d}_1}$.
The shape function spaces and the interpolation operators of the finite elements on $\hat{K}$ are defined by

(2.1)
$$\hat{P}^1 = span\{1,\xi,\eta,\varphi(\xi),\varphi(\eta)\}, \frac{1}{|\hat{K}|}\int_{\hat{K}}(\hat{v}-\hat{I}^1\hat{v})d\xi d\eta = 0, \frac{1}{|\hat{l}_k|}\int_{\hat{l}_k}(\hat{v}-\hat{I}^1\hat{v})d\hat{s} = 0,$$

$$\hat{P}^2 = span\{1,\xi,\eta\}, \frac{1}{|\hat{l}_k|}\int_{\hat{l}_k}\hat{I}^2\hat{v}d\hat{s} = \frac{1}{2}(\hat{v}(\hat{d}_k)+\hat{v}(\hat{d}_{k+1})), \tag{2.2}$$