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L^∞-ERROR ESTIMATES FOR GENERAL OPTIMAL CONTROL PROBLEM BY MIXED FINITE ELEMENT METHODS

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Abstract. In this paper, we investigate the L^∞-error estimates for the solutions of general optimal control problem by mixed finite element methods. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We derive L^∞-error estimates of optimal order both for the state variables and the control variable.

Key Words. L^∞-error estimates, mixed finite element, optimal control.

1. Introduction

Optimal control problems [17] have been extensively utilized in many aspects of the modern life such as social, economic, scientific and eigineering numerical simulation. Due to the wide applications of these problems, they must be solved successfully with efficient numerical methods. Among these numerical methods, finite element discretization of the state equation is widely applied though other methods are also used. There have been extensive studies in convergence of finite element approximation of optimal control problems, see, for example [1], [2], [7], [11], [14], [15], [18], [19] and [28]. A systematic introduction of finite element method for PDEs and optimal control can be found in, for example, [8], [13], [23] and [27].

Many contributions have been done to the L^∞ convergence theory, see [3], [9], [16], [20]. In [3], the authors studied L^∞-error estimates for a semilinear elliptic control problem with standard finite element methods. We also see the earlier work [16] in which L^∞ estimate was obtained for the solution of a semilinear second order elliptic problem by mixed methods. But it didn't focus on optimal control problem. More recently, C. Meyer and A. Rösch have studied the superconvergence property for linear-quadratic optimal control problem in [21], they also investigated the L^∞ estimates with standard finite element for this problem in [20]. Most recently, in [10], the authors studied L^∞-error estimates and superconvergence in maximum norm of mixed finite element methods for NonFickian flows in porous media. However, there doesn't seem to exist much work on theoretical analysis of mixed finite element approximation for optimal control problem in the literature.

In this paper, we will study the L^∞-error estimates for general convex optimal control problem with mixed methods. We have done some primary works on linear-quadratic optimal control problem in which L^∞ estimates for state variables and

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control variable were obtained with mixed methods. Here, in this paper, we will show that also for general convex optimal control problem the similar results can be obtained.

The problem that we will study is the following optimal control problem:

$$\min_{u\in K\subset L^\infty(\Omega)} \left\{ \int_\Omega (g_1(\boldsymbol{p}(x)) + g_2(y(x)) + h(u(x)))dx \right\} \tag{1}$$

subject to the state equation

$$-\mathrm{div}(A\,\mathbf{grad}y) = u, \qquad x \in \Omega, \tag{2}$$

$$y = 0, \qquad x \in \partial\Omega. \tag{3}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p} = u, \qquad x \in \Omega \tag{4}$$

$$\boldsymbol{p} = -A\,\mathbf{grad}y, \qquad x \in \Omega \tag{5}$$

$$y = 0, \qquad x \in \partial\Omega \tag{6}$$

where $\Omega \subset R^2$ is a bounded domain with Lipschitz continuous boundary. Here, $g_1 = g_1(\cdot,\cdot)$, g_2 and h are strictly convex functionals which are continuously differentiable. In the rest of the paper, we shall simply write $g_1(\boldsymbol{p}(x)), g_2(y(x))$ and $h(u(x))$ as $g_1(\boldsymbol{p}), g_2(y)$ and $h(u)$. We further assume that $h(u) \to +\infty$ as $\| u \| \to \infty$. K denotes the admissible set of the control variable, defined by

$$K = \{u \in L^\infty(\Omega) : a \le u \le b \;\; a.e.\ in\ \Omega\}, \tag{7}$$

where a and b are real numbers.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\| \cdot \|_{m,p}$ given by

$$\| \phi \|^p_{m,p} = \sum_{|\alpha|\le m} \| D^\alpha \phi \|^p_{L^p(\Omega)},$$

a semi-norm $| \cdot |_{m,p}$ given by

$$| \phi |^p_{m,p} = \sum_{|\alpha| = m} \| D^\alpha \phi \|^p_{L^p(\Omega)} .$$

We set $W^{m,p}_0(\Omega) = \{\phi \in W^{m,p}(\Omega) : \phi \mid_{\partial\Omega} = 0\}$. For p=2, we denote

$$H^m(\Omega) = W^{m,2}(\Omega), H^m_0(\Omega) = W^{m,2}_0(\Omega),$$

and

$$\| \cdot \|_m = \| \cdot \|_{m,2}, \| \cdot \| = \| \cdot \|_{0,2} .$$

In addition we use $\| \cdot \|_{0,\ \infty}$ to denote the maximum norm in $L^2(\Omega)$.

2. Mixed finite element approximation of optimal control problems

In this section, we study the mixed finite element approximation of the problem (1) and (4)-(6). First, we assume that $A(x) = (a_{ij}(x))$ is a symmetric matrix with $a_{ij}(x) \in W^{1,\infty}(\Omega)$ and for any vector $\boldsymbol{X} \in R^2$, there is a constant $c > 0$, such that

$$\boldsymbol{X}^t A \boldsymbol{X} \ge c \| \boldsymbol{X} \|^2_{R^2} .$$

Next, we introduce the co-state elliptic equation

$$-\mathrm{div}(A(x)(\mathbf{grad}z + g_1'(\boldsymbol{p}))) = g_2'(y), \qquad x \in \Omega, \tag{8}$$

with the boundary condition

$$z = 0, \qquad x \in \partial\Omega. \tag{9}$$