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FINITE ELEMENT APPROXIMATION FOR TV REGULARIZATION

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Abstract. In this paper, we will develop the convergence of the solution of TV-regularization equations with regularized parameter $\varepsilon \longrightarrow 0$ in $BV(\Omega)$ for practical purposes. Originated from the effects of regularized parameter ε, the error rate of finite element approximation for TV-regularization equations will be controlled by the regularized parameter ε^{-1} polynomially in the energy norm when using linearization technique and duality argument. And in the L^p-norm, the effect of regularized parameter ε will be more extremely.

Key Words. TV-regularization, Regularized Parameter, Finite Element Method

1. Introduction

We consider the following total variation(TV) regularization equations

$$div\big(\frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}}\big)-\lambda(u^\varepsilon-g)=0, \quad in\ \Omega, \tag{1}$$

$$\frac{\partial u^\varepsilon}{\partial n}=0, \quad on\ \partial\Omega. \tag{2}$$

Equation (1) is an Euler-Lagrange equation originated from the following unconstraint minimization problem

$$\min_{u^\varepsilon} J_{\lambda,\varepsilon}(u^\varepsilon)=\min_{u^\varepsilon}\{\int_\Omega \sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}dx+\frac{\lambda}{2}\int_\Omega |u^\varepsilon-g|^2dx\}. \tag{3}$$

Usually, equations (1)-(2) are the numerical regularized approximation of the following equations, respectively

$$div\big(\frac{\nabla u}{|\nabla u|}\big)-\lambda(u-g)=0, \quad in\ \Omega, \tag{4}$$

$$\frac{\partial u}{\partial n}=0, \quad on\ \partial\Omega. \tag{5}$$

which corresponds to an unconstrained minimization problem

$$\min_{u} J_{\lambda}(u)=\min_{u}\{\int_\Omega |\nabla u|dx+\frac{\lambda}{2}\int_\Omega |u-g|^2dx\}. \tag{6}$$

where, especially in image processing, $\lambda>0$ is the penalization parameter which controls the trade-off between goodness of fit-to the data and variability in u, $u:\Omega\subset\mathcal{R}^2\longrightarrow\mathcal{R}$ denote the gray level of an image describing a real scene, and g be the observed image of the same scene, which is a degradation of u. And (6) is usually called the total variation (TV) model or ROF model duo to Rudin, Osher

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and Fatemi [20]. It is one of the best known and most successful noise removal and image restoration model, too.

(1)-(2) can be taken as nonlinear elliptic problem and we will consider finite element method approximation in this paper. The common nonlinear elliptic problem, with Dirichlet boundary conditions , Neumann boundary conditions or mixed boundary conditions, have been studied theoretically and numerically in the past thirty years, see [24, 19, 18, 12, 4, 11, 10, 2, 13, 14]. In [2], the authors proved the existence and uniqueness of the solution of nonlinear elliptic equations of monotone type and also derived error estimates for the finite element approximation in the energy norm as well as assumption that $u \in W_p^1, p > 1$. In [10], they presented Galerkin approximations of a quasi-linear non-potential elliptic problem of a non-monotone type. For the $u \in H^1(\Omega)$, u_h converges to u weakly and for the $u \in W_p^1(\Omega), p > 2$, u_h converges to u strongly in the H^1-norm. In, [11], the existence and uniqueness of the finite element solution of quasi-linear elliptic equations with mixed Dirichlet-Neumann conditions are derived by developing a one-parameter family of $hp-$version discontinuous Galerkin finite element methods in the divergence form on a bounded open set Ω. If $\lambda = 0$, (4)-(5) or (1)(2) can be regards as the mean curvature problems. In [12, 7], numerical approximation for the mean curvature was also set up on finite element error estimations and adaptive algorithm, respectively.

Interestingly, responding to (6), it is usually solved by formulating the steepest descent gradient method, which motivates to consider its gradient flow as well as its numerical form like (1)(2). In [5, 6], they considered the relations between $u(t)$ and $u^\varepsilon(t)$, proved that $u^\varepsilon(t)$ converged to $u(t)$ in $L^1((0,T);BV(\Omega))\bigcap C^0([0,T],L^2(\Omega))$. Dramatically, the convergence rate of the finite element approximation is depend on the parameter ε by the form $C(\frac{1}{\varepsilon})$. It is important for such a result when we deal with the similar numerical problems because we have to select a proper mesh size to keep the convergence by finite element method or finite difference method.

Numerically, some works have pointed out that the chose of regularized parameter ε is vital in image processing, see [3, 22, 23]. The selection of an appropriate regularized parameter has been one of difficulties in image processing. Some others have pointed out that ε will influence the convergence rate of level set function, for example, in inverse problems, [16, 15, 17]. Therefore, one of the aims of this paper is to construct and analyze a finite element method for approximating the solution of equation (1)-(2) for each $\varepsilon > 0$ and approximating the solution of equation (4)-(5) by taking $\varepsilon \longrightarrow 0$.

Based on the above discussing, in this paper, our presentation follows the frameworks established in [10, 11, 5, 6] in order to develop the convergence relation of u, u^ε in the space $BV(\Omega)$, and error convergence rate of finite element approximation for u^ε. And we also try to demonstrate how ε affects the convergence rate of finite element approximation u^ε.

This paper is organized in the following way. In section §2, we prove that the solution u^ε of problem (1)-(2) will converge to the solution u of problem (4)-(5) in $BV(\Omega)$ space when the regularized parameter $\varepsilon \longrightarrow 0$. In section §3, by introducing the linearization of the nonlinear problem, we give coercion and duality operator of the linearization operator, which are the foundation of studying the nonlinear elliptic partial differential equation for finite element methods. In section §4, firstly, we introduce an operator T which is contract proved by Lemma 3. Then, based on fixed point theorem, we prove that the fixed point of operator T is the solution of finite element approximating for the variation of problem (1)-(2) in the energy