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SUBSTRUCTURING PRECONDITIONERS FOR PARABOLIC PROBLEMS BY THE MORTAR METHOD

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Abstract. We study substructuring preconditioners for the linear system arising from the discretization of parabolic problems when the mortar method is applied. By using a suitable non standard norm equivalence we build an efficient edge block preconditioner and we prove a polylogarithmic bound for the condition number of the preconditioned matrix.

Key Words. Domain decomposition, iterative substructuring, mortar methods, parabolic equations.

1. Introduction

We deal with the efficient construction of preconditioners for the linear system associated to the discretization of parabolic problems when a domain decomposition method is applied. Different domain decomposition methods for parabolic problems can be found in literature, see e.g. [14, 11, 12, 25] but here we focus on the mortar method which is a nonconforming domain decomposition method that allows different discretization and/or methods in different subdomains and that weakly enforces the matching of discretizations on adjacent subdomains (see [3, 4, 8, 23]).

Implicit schemes in the time variable, such as the backward Euler and Crank-Nicolson, are considered hence, at a fixed time level, we have to solve an elliptic problem depending on the time step parameter. Consequently, we might apply the methods originally proposed for elliptic equations (see [13, 24, 21, 22]) but here we propose a preconditioner that takes into account the parabolic structure of the original problem. More specifically, after elimination of the degrees of freedom internal to the subdomains, we have to find the traces of the solution on the subdomain boundaries, i.e. to solve the Schur complement system. The approach considered here is the substructuring one, proposed in [9] for conforming domain decomposition and already applied to the mortar method in [1] for the case of order one finite elements and then generalized to a general class of discretization spaces in [7, 6]. A suitable splitting of the nonconforming discretization space in terms of "edge" and "vertex" degrees of freedom is considered and then the related block-Jacobi type preconditioners are used.

In order to design a convenient and inexpensive preconditioner, the edge and vertex blocks have to be replaced in a suitable way; indeed they are not explicitly constructed but it is important to compute efficiently the action of their inverse. For elliptic problems an efficient approximation of the edge block was built by using a norm equivalence for the space $H^{1/2}_{00}$ (see [9]). Analogously here, we propose an equivalent but cheaper to implement edge block preconditioner for parabolic

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problems by proving a suitable non standard norm equivalence. We show that the edge block can be built by adding to the known preconditioners for elliptic problems a new term that can be easily computed and that was suggested by the norm equivalence proved.

Following the abstract formulation presented in [7, 6] we prove that the condition number of the preconditioned matrix grows at most polylogarithmically with the number of degrees of freedom per subdomain, analogously to what happens for the elliptic case and it remains bounded independently of the time step parameter.

The outline of the paper is the following. In sections 2 and 3 we introduce the parabolic problem and we briefly review the mortar method and its main properties. In section 3 we define suitable norms for the trace space that will be crucial for the construction of the preconditioner. The substructuring preconditioner is proposed and studied in section 4. The main theorem of the paper (Theorem 4.1) stating the convergence of the method and the polylogarithmic bound for the condition number of the preconditioned matrix is presented in the same section. Numerical experiments that validate the theory are shown in Section 5. Finally, to help the reader, the Appendix collects some lemmas used in the paper.

For convenience, the symbols $\lesssim$, $\gtrsim$ and $\simeq$ will be used in the paper, i.e. $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \simeq y_3$ mean that $x_1 \leq c_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants c_1, c_2, c_3, C_3 independent of the mesh and time step parameters.

2. A parabolic problem

We consider the following parabolic problem:

find $u(x,t)$ *such that*:

$$\begin{cases} \dfrac{\partial u}{\partial t} - div(A(\mathbf{x})\nabla u) = f & \text{in } \Omega\times]0,T[\\ u(x,t) = 0 & \text{on } \partial\Omega\times]0,T[\\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with boundary $\partial\Omega$, $f \in L^2(\Omega)$ and the matrix $A(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1,2}$ is assumed to be, for almost all $\mathbf{x} \in \Omega$, symmetric positive definite with smallest eigenvalue $\geq \alpha > 0$ and largest eigenvalue $\leq \alpha'$, α, α' independent of $\mathbf{x}$. The weak formulation of Problem (1) is:

for $t \in]0,T[$, *find* $u(x,t) \in H_0^1(\Omega)$, $u(x,0) = u_0(x)$ *in* Ω, *such that*

$$\left(\frac{\partial u}{\partial t}, v\right) + a(u,v) = (f,v),$$

with the bilinear form $a(\cdot,\cdot)$ defined as

$$a(u,v) := \sum_{i,j} a_{ij}(\mathbf{x}) \frac{\partial u}{\partial \mathbf{x}_i} \frac{\partial v}{\partial \mathbf{x}_j}\, d\mathbf{x} \tag{2}$$

assumed to be bounded and elliptic and the linear functional

$$(f,v) = \int_\Omega f v\, dx.$$

We consider two types of time discretization, namely, the backward Euler scheme and the Crank-Nicolson scheme. Both scheme are absolutely stable (see [18]). Let τ_n be the n-th time step, then the two schemes lead to the following problems:

for a given $g \in L^2(\Omega)$, *find* $u \in H_0^1(\Omega)$ *such that*

$$(u,v) + \tau a(u,v) = (\tau g, v), \qquad \forall v \in H_0^1(\Omega) \tag{3}$$