INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 4, Pages 612–634

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Computing and Information

THE HOLE-FILLING METHOD AND THE UNIFORM MULTISCALE COMPUTATION OF THE ELASTIC EQUATIONS IN PERFORATED DOMAINS

XIN WANG AND LI-QUN CAO

Abstract. In this paper, we discuss the boundary value problem for the linear elastic equations in a perforated domain Ω^ε. We fill all holes with a very compliant material, then we study the homogenization method and the multiscale analysis for the associated multiphase problem in a domain Ω without holes. We are interested in the asymptotic behavior of the solution for the multiphase problem as the material properties of one weak phase go to zero, which has a wide range of applications in shape optimization and in 3-D mesh generation. The main contribution obtained in this paper is to give a full mathematical justification for this limiting process in general senses. Finally, some numerical results are presented, which support strongly the theoretical results of this paper.

Key Words. homogenization, multiscale analysis, elastic equations, perforated domain, hole-filling method.

1. Introduction

In this paper, we consider the following boundary value problems of elastic equations in a perforated domain:

$$
(1) \qquad \left\{
\begin{array}{l}
-\dfrac{\partial}{\partial x_j}(a_{ijkh}(\frac{x}{\varepsilon})\dfrac{\partial u_k^\varepsilon(x)}{\partial x_h}) = f_i(x), \quad i = 1, 2, \cdots, n, \text{ in } \quad \Omega^\varepsilon \\
\sigma_\varepsilon(u^\varepsilon) = 0, \quad \text{on} \quad S_\varepsilon \\
u^\varepsilon(x) = g_0(x), \quad \text{on} \quad \Gamma_1 \\
\sigma_\varepsilon(u^\varepsilon) = g_1(x), \quad \text{on} \quad \Gamma_2
\end{array}
\right.
$$

Following Oleinik's notation (see [22]), let $Q = \{\xi : 0 < \xi_j < 1, j = 1, \cdots, n\}$, and ω be an unbounded domain of R^n which satisfies the following conditions:

(B_1) ω is a smooth unbounded domain of R^n with a 1-periodic structure.

(B_2) The cell of periodicity $\omega \cap Q$ is a domain with a Lipschitz boundary.

(B_3) The set $Q\backslash\bar{\omega}$ and the intersection of $Q\backslash\bar{\omega}$ with the δ_0-neighborhood ($\delta_0 < \frac{1}{4}$) of ∂Q consist of a finite number of Lipschitz domains separated from each other and from the edges of the cube Q by a positive distance.

Suppose that Ω^ε is a domain which has the form: $\bar{\Omega}^\varepsilon = \bar{\Omega}_0^\varepsilon \cup (\bar{\Omega}\backslash\Omega_0)$, where Ω is a bounded Lipschitz convex domain of R^n without holes, $\bar{\Omega}_0 = \cup_{z\in T_\varepsilon}\varepsilon(z+\bar{Q}) \subset \Omega$, $\bar{\Omega}_0^\varepsilon = \bar{\Omega}_0 \cap \varepsilon\bar{\omega}$ is shown in Fig.1(a), T_ε is the subset of Z^n consisting of all z such that $\varepsilon(z+Q) \subset \Omega$. The domain $\Omega_1 = \Omega\backslash\bar{\Omega}_0$ denotes the boundary layer as shown in Fig.1(b). The boundary $\partial\Omega^\varepsilon$ of a perforated domain Ω^ε is composed of $\partial\Omega$ and

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Received by the editors August 10, 2007 and, in revised form, November 5, 2007.

2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.

This work is supported by National Natural Science Foundation of China (grant # 90405016), and Special Funds for Major State Basic Research Projects(grant # 2006CB605102).

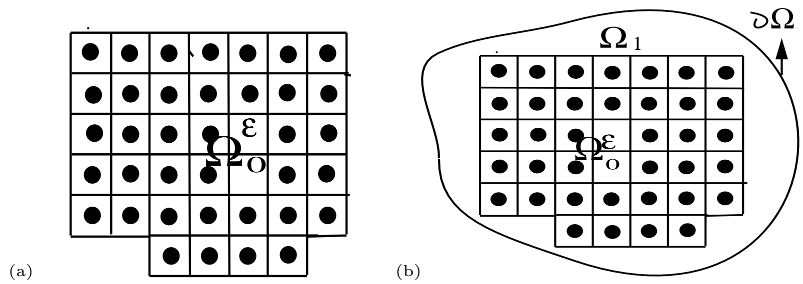

FIGURE 1. : (a) interior domain Ω_0^ε; (b) boundary layer.

the surfaces S_ε of cavities, where $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2, \Gamma_1 \cap \Gamma_2 = \emptyset$. Such a domain Ω^ε is called as a type-II domain (see, [22]).

In equations (1), $u^\varepsilon(x) = (u_1^\varepsilon(x), \cdots, u_n^\varepsilon(x))^T$ denotes a displacement function, $f(x) = (f_1(x), \cdots, f_n(x))^T$ is a body force, $g_0(x)$ is a given displacement function on the Dirichlet boundary Γ_1, $g_1(x)$ is a given surface stress on the Neumann boundary Γ_2, $\sigma_\varepsilon(u^\varepsilon) = (\sigma_{\varepsilon,1}(u^\varepsilon), \cdots, \sigma_{\varepsilon,n}(u^\varepsilon))$, $\sigma_{\varepsilon,i}(u^\varepsilon) \equiv \nu_j a_{ijhk}^\varepsilon \dfrac{\partial u_h^\varepsilon}{\partial x_k}$, $i = 1, \cdots, n$, where $\vec{n} = (\nu_1, \cdots, \nu_n)$ is the unit outer normal vector to $\partial\Omega^\varepsilon = \partial\Omega \cup S_\varepsilon$.

Suppose that

(A_1) Let $\xi = \varepsilon^{-1}x$, and the elements of a matrix $(a_{ijkh}(\xi))$ are 1-periodic functions in ξ.

(A_2) $a_{ijkh}(\xi) = a_{jikh}(\xi) = a_{khij}(\xi)$.

(A_3) $\gamma_0\eta_{ij}\eta_{ij} \le a_{ijkh}(\xi)\eta_{ij}\eta_{kh} \le \gamma_1\eta_{ij}\eta_{ij}$, $\xi \in \omega$, $\gamma_0, \gamma_1 > 0$, where (η_{ij}) is any real symmetric matrix.

(A_4) $a_{ijkh} \in L^\infty(\omega)$, $f \in L^2(\Omega^\varepsilon)$, $g_0 \in H^{\frac{1}{2}}(\Gamma_1)$, $g_1 \in L^2(\Gamma_2)$.

Remark 1.1. *Existence and uniqueness of the solution to problem (1) can be established on the basis of the assumptions* $(B_1) - (B_3)$*, and* $(A_1) - (A_4)$ *(see, e.g.* [22]*).*

The numerous studies of homogenization and its applications for problem (1) in perforated domains containing many small holes have been developed by so many contributions that it is impossible to quote them all. We refer the interested reader to these books and articles (see, e.g. [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 14, 16, 17, 18, 19, 20, 21, 22]).

When we solve the elastic equations with homogeneous Neumann boundary conditions on the surfaces of holes in a perforated domain, we usually fill these holes with an almost degenerated phase, which is also called the hole-filling method. Actually, engineers often use the method to predict the effective properties of perforated materials. From a physical point of view, when the material properties of the weak phase go to zero, this limit procedure is clear. But a full mathematical justification has not been seen in all the available literature. In this paper, we try to give a full mathematical justification for this limiting process in general cases. Furthermore, in order to compute the displacement and the stress field in a domain, we present a uniform multiscale method for solving the elastic equations (1) regardless of whether there are holes or not. The crucial step of the method is to define the cell functions which are different from those of classical homogenization method. On the other hand, from the viewpoint of numerical computation, the mesh generation in a 3-D perforated domain is somehow more difficult than that