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AN ALTERNATING DIRECTION GALERKIN METHOD COMBINED WITH A MODIFIED METHOD OF CHARACTERISTICS FOR MISCIBLE DISPLACEMENT INFLUENCED BY MOBILE AND IMMOBILE WATER

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Abstract. Numerical approximations are considered for a mathematical model for miscible displacement influenced by mobile and immobile water. A mixed finite element method is adopted to give a direct approximation of the velocity, the concentration in mobile water is approximated by an alternating direction Galerkin finite element method combined with the method of characteristics and the concentration in immobile water is approximated by a standard Galerkin method. Optimal order L^2- and H^1-error estimates are derived.

Key Words. alternating direction Galerkin method, modified method of characteristic, finite element methods, error estimate.

1. Introduction

In recent years, the intentional or accidental release of chemical wastes on soils has further stimulated current interests in the movement of chemicals. Displacement studies have become important tools in soil physics, particularly for predicting the movement of pesticide, nitrates, heavy metals, and other solutes through soils.

The soil structure is complex. In aggregated media, soils are composed of slowly and quickly conducting pore sequences, the liquid-filled and dead-end pores; or immobile water exists. In [12] the movement of a chemical through a sorbing porous medium with a lateral or intra-aggregated diffusion was considered. The liquid in porous media is divided into mobile and immobile regions. Mobile water is located inside the larger pores. The flow in mobile water is assumed to occur in this region only. Solute transfer in mobile water occurs by both convection and longitudinal diffusion. Immobile water is located inside aggregates and at the contact points of aggregates and/or particles. Diffusion transfer between the two liquid regions is assumed to be proportional to the concentration difference between the mobile and immobile liquids. A dynamic soil region is located sufficiently close to the mobile water phase for equilibrium between the solute in the mobile liquid and that sorbed by this part of the soil mass. A stagnant soil region, where sorption is diffusion limited, is located mainly around the micro-pores inside the aggregates or along dead-end water pockets. Sorption occurs here only after the chemicals have diffused through the liquid barrier of the immobile liquid phase. In [12] sorption

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process in both the dynamic and stagnant regions of the medium was assumed to be instantaneous and the adsorption isotherm was assumed to be linear.

The Darcy velocity of the fluid mixture is given by [2, 3, 4]

$$\mathbf{u} = -a(c)\nabla p \, , \tag{1.1}$$

where $a(c) = \kappa(x)/\nu(c)$, κ is the permeability of the medium, ν the concentration-dependent viscosity, and p the pressure. Incompressibility implies

$$\nabla \cdot \mathbf{u} = q, \tag{1.2}$$

where $q = q(x,t)$ is the imposed external flow. The equation for the concentration can be put in the form [2, 12, 14]:

$$s_1 \frac{\partial c}{\partial t} + s_2 \frac{\partial c'}{\partial t} + \mathbf{u} \cdot \nabla c - \nabla \cdot (D \nabla c) = q(c^* - c), \quad x \in \Omega, \quad t \in J, \tag{1.3}$$

$$\frac{\partial c'}{\partial t} = \alpha(c - c'), \quad x \in \Omega, \quad t \in J, \tag{1.4}$$

where $J = [0,T]$, $s_1(x) = (\theta_m + f\rho K)/\theta_m, s_2(x) = (\theta_{im} + (1-f)\rho K)/\theta_{im}$, and $\alpha = \alpha_0/(\theta_{im} + (1-f)\rho K)$; f is the fraction adsorption in dynamic region; K the constant in the Freundlich isotherm; θ_m and θ_{im} the mobile water and immobile water content, respectively; ρ the bulk density; c and c' denote the solute concentrations in the mobile water and immobile water regions, respectively; D the dispersion coefficient, α_0 the mass exchange coefficient; c^* the concentration of the contamination.

The boundary and initial conditions can be imposed in the following form:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad D\nabla c \cdot \mathbf{n} = 0, \quad x \in \partial\Omega, \quad t \in J, \tag{1.5}$$

$$c(x,0) = c_0(x), \quad c'(x,0) = c_0'(x), \quad x \in \Omega, \tag{1.6}$$

where $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$ of the domain Ω. For compatibility one requires that

$$\int_\Omega q(x,t)dx = 0, \qquad \text{for all} \;\; t \in J.$$

Our objective is to design and analyze a numerical method for approximating the solution of the system (1.1)–(1.4) subject to the initial and boundary conditions (1.5) and (1.6). Note that the pressure does not appear explicitly in the equation (1.3) for concentration; however, velocity does. A mixed finite element method will be adopted here to approximate the pressure p and the velocity $\mathbf{u}$ simultaneously. The concentration c' in the immobile water will be approximated using a standard Galerkin finite element method. The concentration c in the mobile water will be approximated by an alternating direction method combined with the method of characteristics which combines the attractive attributes of the two methods.

In 1971, Douglas and Dupont in [9] formulated a Galerkin alternating-direction procedures for nonlinear parabolic equations posed on a rectangular region with a uniform grid. The alternating-direction method reduces multidimensional problems to a collection of one-dimensional problems and the matrices that must be inverted at each time step of the solution process are independent of time and require only one decomposition. The storage requirements for these matrices are associated with one-dimensional problems rather than the full multi-dimensional problem, so the storage requirements can be quite low. The Galerkin alternating-direction method is particularly attractive for solving large three-dimensional nonlinear problems. A survey of some results in the use of the alternating-direction finite element methods