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A NOTE ON THE APPROXIMATION PROPERTIES OF THE LOCALLY DIVERGENCE-FREE FINITE ELEMENTS

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Abstract. This paper investigates construction and approximation properties of the locally divergence-free (LDF) finite elements. Numerical stability of the natural and normalized bases for the LDF elements is analyzed. Error estimates about the jumps and the total divergence of the localized $\mathbf{L}_2$-projection are proved and validated through numerical examples.

Key Words. approximation property, locally divergence-free (LDF), localized $\mathbf{L}_2$-projection, solenoidal

1. Introduction

"Divergence-free" is an important physical property that appears in many applications, for example, incompressible fluid flows and solenoidal magnetic fields. The divergence-free property should be preserved by numerical methods, globally or locally, in the classical or weak sense [7]. An early study on magnetohydrodynamics (MHD) [4] has shown that numerical errors in the divergence of a magnetic field may build up in time and bring up nonphysical phenomena in numerical simulations, for instance, loss of momentum and energy conservation.

The locally divergence-free finite elements [1] have regained researchers' interests in recent years. The LDF elements are devised to preserve the divergence-free property locally or pointwise inside each element. These elements typically have polynomial shape functions. When the LDF elements are glued together to form an approximation subspace, continuity across element interfaces is usually lost. Continuity can be enforced in the normal directions of element interfaces to construct nonconforming LDF finite element approximation subspace. This approach is adopted in [5] for solving the reduced time-harmonic Maxwell equations. Another approach is to place the LDF finite elements in the framework of the discontinuous Galerkin methods and weak discontinuity is then enforced by penalty factors. Applications along this line can be found in [1, 10, 11] for solving stationary Stokes and Navier-Stokes problems, [8] for solving the Maxwell equations, and [13] for solving the ideal MHD problem.

For implementations of the LDF finite elements, the natural basis functions [1] are conceptually simple and appealing, but their divergence-free property is not preserved by affine mappings. In this paper, we will show that the natural basis functions are actually numerically unstable, so normalization on shape functions should be adopted. When the LDF elements are used in the discontinuous Galerkin framework, there are jumps across the element interfaces. In other words,

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the "total divergence" of a numerical solution might not be zero. The jumps or the total divergence in LDF finite element approximations need to be measured. Especially, when the LDF elements are used with the characteristic methods [14], the foot/head of a characteristic might fall right on the interface of two elements. The localized $\mathbf{L}_2$-projection [3] could be used to approximate the initial solenoidal vector fields for time-dependent problems. How good could the localized $\mathbf{L}_2$-projection be (regarding the jumps or the total divergence)? This paper addresses the above issues. In Section 2, we show that the natural bases for the LDF elements are numerically unstable and propose some normalized bases for the LDF elements. Section 3 discusses the approximation properties of the LDF elements with a focus on the localized $\mathbf{L}_2$-projection. The theoretical error estimates are illustrated and validated in Section 4 through a very smooth vector field and a nonsmooth field.

Throughout the paper, we shall use $A \lesssim B$ to represent an inequality $A \leq CB$, where C is a generic positive constant independent of the mesh size.

2. Natural and Normalized Bases for the LDF Elements

"Locally divergence-free" is actually a pointwise property, unrelated to the geometric shapes of finite elements. It can be proved that for a two-dimensional vector field $\mathbf{v}$, it is divergence-free ($\operatorname{div} \mathbf{v} = 0$) if and only if there exists a scalar potential function $A(x, y)$ such that

$$\mathbf{v}(x, y) = \mathbf{curl}\, A = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x} \right).$$

Therefore, one can take the curl of the natural basis polynomials

$$1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^3,\ x^2y,\ xy^2,\ y^3,\ \ldots$$

to get a natural basis for the LDF finite elements. Clearly, there are 2 zeroth-order and 3 first-order basis functions:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} x \\ -y \end{pmatrix}. \tag{1}$$

There are 4 second-order and 5 third-order basis functions:

$$\begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} x^2 \\ -2xy \end{pmatrix}, \begin{pmatrix} -2xy \\ y^2 \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix}, \begin{pmatrix} x^3 \\ -3x^2y \end{pmatrix}, \begin{pmatrix} -3xy^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} x^2y \\ -xy^2 \end{pmatrix}. \tag{3}$$

Three-dimensional natural bases for the LDF finite elements can be constructed similarly [1].

Although theoretically these basis functions can be used for any elements, they are numerically unstable. The mass/Gram matrix of these basis functions could be very ill-conditioned. For example, suppose we have a uniform rectangular mesh on the unit square $[0, 1] \times [0, 1]$ with a mesh size 0.01 in each direction. If we use all these natural basis functions up to order 3 on the rectangle element $[0, 0.01] \times [0, 0.01]$, then the condition number of the mass matrix measured in 2-norm is as high as 2.800×10^{15}. This could result in unpredictable round-off errors that fail the Cholesky factorization process in the localized $\mathbf{L}_2$-projection, to be discussed later in Section 3.

Therefore, some sort of local basis functions are needed for different elements. However, we may not use the affine mappings from generic elements to the reference