INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 4, Pages 729–748

CONSTRUCTION OF BOUNDARY LAYER ELEMENTS FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATIONS AND L^2- STABILITY ANALYSIS

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Abstract. It has been demonstrated that the ordinary boundary layer elements play an essential role in the finite element approximations for singularly perturbed problems producing ordinary boundary layers. Here we revise the element so that it has a small compact support and hence the resulting linear system becomes sparse, more precisely, block tridiagonal. We prove the validity of the revised element for some singularly perturbed convection-diffusion equations via numerical simulations and via the H^1- approximation error analysis. Furthermore due to the compact structure of the boundary layer we are able to prove the L^2- stability analysis of the scheme and derive the L^2- error approximations.

Key Words. boundary layer, boundary layer element, finite elements, singularly perturbed problem, convection-diffusion, stability, enriched subspaces, exponentially fitted splines.

1. Introduction

In this article we consider linear singularly perturbed boundary value problems of the types:

$$-\epsilon\triangle u^\epsilon - u^\epsilon_x = f \text{ in } \Omega, \tag{1.1a}$$

$$u^\epsilon = 0 \text{ on } \partial\Omega, \tag{1.1b}$$

where $0 < \epsilon << 1,\ \Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$. The function f is assumed to be smooth on $\bar{\Omega}$ but only in Section 3 below we will assume (for the L^2- stability analysis) that f belongs to $L^2(\Omega)$. Problem (1.1) is meant to be a simplified model for a class of problems involving variable coefficients and curved boundaries. However the treatment of these more involved problems only involve additional purely technical difficulties and we thought it would be more appropriate to present our results in the case of this model problem. Variable coefficients equations, curved boundaries and other generalizations will be addressed in separate works.

As ϵ becomes small, the solutions to problem (1.1) generally display, near the boundaries, thin transition layers called boundary layers, which are due to the fact that the boundary conditions of the problem are not the same for $\epsilon > 0$ and $\epsilon = 0$, and then (for $\epsilon > 0$ small) certain derivatives of the solutions become very large near the boundaries. We expect that within these boundary layers, the approximation errors of the discretized system corresponding to problem (1.1)

Received by the editors November 15, 2007 and, in revised form, January 23, 2008.
2000 *Mathematics Subject Classification.* 65N30, 34D15, 76N20.
This work was supported in part by NSF-DMS grants 0305110 and 0604235, and by the Research Fund of Indiana University.

become very large (due to the large H^2- singularities of the boundary layers). When the stiffness of the discretized systems is not properly handled, those large approximation errors at the boundaries propagate in the whole domain due to the convective term, e.g. $-u_x$ in (1.1a), and then the numerical solutions show a highly oscillatory behavior, see e.g. [20], [22], [3], [4], [13], [14] and [15]. Resolving boundary layers by the classical approximation methods requires very fine meshes, which is costly to realize in practice. Indeed, the thickness of the boundary layers (of order $O(\epsilon)$ for ordinary boundary layers (OBL), and of order $O(\epsilon^{1/2})$ for parabolic boundary layers (PBL), see [23], [15]) is usually much smaller than the mesh size h. Notice that our problem (1.1) produces both OBLs at $x = 0$ and PBLs at $y = 0, 1$, which pollute the numerical solutions, globally and locally respectively. In view of properly approximating such problems, it has been suggested by Han and Kellogg, in [10], [11] to add to the Galerkin space suitable profile functions encompassing the main features of the boundary layers, leading to the so-called *enriched subspaces* (ES) method. In this article and related ones [3], [4] we call Boundary Layer Elements (BLE) these profile functions. A related concept is that of *exponentially fitted splines (or L- splines)* (EFS) where the Galerkin basis of spline functions is chosen (constructed) adapted to the operator L_ϵ; see [9] for one-dimensional two-point boundary value problems and [6], [7] and [18] for two-dimensional ones. Our works is closer to the enriched subspaces point of views, and we use asymptotic expansions inspired in part by the work [23] to construct the boundary layer elements using asymptotic expansion techniques. We were not aware of this series of articles on enriched subspaces and exponentially fitted splines when we started our own work in [3], [4], [13] - [16]. Comparisons between these articles and our own past and current work are made below.

Before we proceed, we introduce the notations, the semi-norms and norms for the Sobolev spaces $H^m(\Omega)$, $m \geq 0$ integer (for $m = 0$, it is denoted L^2), which are, respectively, $|u|_{H^m} = \left\{ \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^2 d\Omega \right\}^{1/2}$ and $\|u\|_{H^m} = \left\{ \sum_{j=0}^m |u|^2_{H^j} \right\}^{1/2}$. The corresponding inner products are $(u, v) = \int_\Omega uv d\Omega$ for L^2, $(\!(u, v)\!) = (u, v) + \int_\Omega \nabla u \cdot \nabla v d\Omega$ for H^1, and $(\!(u, v)\!)_{H^m} = \sum_{|\alpha|\leq m} (D^\alpha u, D^\alpha v)$ for H^m, $m \geq 2$. For the Dirichlet boundary value problem (1.1), we use the Sobolev space $H^1_0(\Omega)$, which is the closure in the space $H^1(\Omega)$ of C^∞ functions compactly supported in Ω. Thanks to the Poincaré inequality the space $H^1_0(\Omega)$ is equipped with the inner product $(\!(\cdot, \cdot)\!) = \int_\Omega \nabla u \cdot \nabla v d\Omega$, and the norm $\| \cdot \| = | \cdot |_{H^1}$.

In [3], [4], [13], [14] and [15], it is demonstrated that the boundary layer elements (BLE), i.e.

$$\phi_0^*(x) = -e^{-x/\epsilon} - (1 - e^{-1/\epsilon})x + 1, \tag{1.2}$$

play an essential role in the finite element approximations for singularly perturbed problems producing the OBLs.

The present article is concerned with two dimensional extensions of [3] and the efficient application of the BLE ϕ_0^*. To solve the problem (1.1) in the finite element context, we consider its weak formulation: *To find* $u \in H^1_0(\Omega)$ *such that*

$$a_\epsilon(u, v) := \epsilon(\!(u, v)\!) - (u_x, v) = (f, v), \ \forall v \in H^1_0(\Omega), \tag{1.3}$$

and then we look for an approximate solution $u_h \in V_h$ such that

$$a_\epsilon(u_h, v_h) = (f, v_h), \ \forall v_h \in V_h, \tag{1.4}$$

where the finite element space V_h will be specified in Section 2.2 below. It contains a classical Q_1 finite element space enriched by a boundary layer element related to ϕ_0^*.