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NUMERICAL METHODS FOR UNSATURATED FLOW WITH DYNAMIC CAPILLARY PRESSURE IN HETEROGENEOUS POROUS MEDIA

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Abstract. Traditional unsaturated flow models use a capillary pressure-saturation relationship determined under static conditions. Recently it was proposed to extend this relationship to include dynamic effects and in particular flow rates. In this paper, we consider numerical modeling of unsaturated flow models incorporating dynamic capillary pressure terms. The resulting model equations are of nonlinear degenerate pseudo-parabolic type with or without convection terms, and follow either Richards' equation or the full two-phase flow model. We systematically study the difficulties associated with numerical approximation of such equations using two classes of methods, a cell-centered finite difference method (FD) and a locally conservative Eulerian-Lagrangian method (LCELM) based on the finite difference method. We discuss convergence of the methods and extensions to heterogeneous porous media with different rock types. In convection-dominated cases and for large dynamic effects instabilities may arise for some of the methods while those are absent in other cases.

Key Words. unsaturated flow, Richards' equation, two-phase flow model, dynamic capillary pressure, pseudo-parabolic equation, finite difference method, locally conservative Eulerian-Lagrangian method, implicit time-stepping

1. Introduction

The main interest of this paper is in numerical algorithms for unsaturated flow in highly heterogeneous media and in particular handling dynamic capillary pressure.

Unsaturated preferential flow in porous media is a physical phenomenon occurring in heterogeneous soils and bedrock and is related to the presence of special features of the medium such as cracks, fissures, and macropores. Such heterogeneities are represented in partial differential equation (PDE) models of the flow by a variation of nonlinear *rock properties* of the medium with position, called the *rock type* dependence. Here we are concerned mainly with the capillary pressure function; that is, the pressure-saturation relationship $S \mapsto P_c(S)$ which, when this property is rock type dependent, it reads $S \mapsto P_c(\mathbf{x}, S)$, where $\mathbf{x}$ denotes position. It is standard practice to include rock type dependence in a reservoir simulator [47]; however, there are few associated mathematical and numerical analyses except [18, 33] and those for multiscale heterogeneities developed in [28, 19, 16, 17].

Additional phenomena occurring in preferential flow such as nonequilibrium effects, hysteresis, and/or large flow velocities have been recently discussed by experimental and theoretical soil physicists. In particular, it has been observed and reconfirmed recently, see [66] and references therein, that rock properties measured

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in a laboratory in equilibrium conditions bear little resemblance to those observed in experiments in case of large fluxes (velocities); it has been in fact postulated that the data collected especially for the capillary pressure has a non-unique character when considered over a range of nonequilibrium conditions. This suggests existence of a hidden variable as pointed out in [41] and can be explained in terms of the imbibition (increasing S) and drainage (decreasing S) hysteresis. Another recently proposed class of model modifications has been the use of *dynamic capillary pressure* [42, 68, 67, 66, 12, 11] which accounts for the dynamic flow rates via $S \mapsto P_c(\mathbf{x}, S, \frac{dS}{dt})$. Some authors believe that dynamic capillary pressure terms could explain instabilities in gravity-driven flow and in particular the phenomena of fingering [51]. Finally, there is evidence that in the presence of strong heterogeneities the conditions at the interface of different rock types should reflect non-equilibrium [55, 19].

Our interest is in numerical modeling of preferential flow in porous media which may occur at more than one scale through macropores or due to small or large inhomogeneities as well as in rock fractures, gravel filled excavated areas etc. Therefore, it is necessary for us to explore the numerical algorithms for dynamic capillary pressure and multiple rock types.

From a theoretical PDE point of view, Richards' model of unsaturated flow is a nonlinear degenerate parabolic equation in the unknown water saturation S; its character depends on the nonlinear diffusion parameter $\mathbf{D}(S)$ which may become degenerate (zero or very large) for some values of S. In addition, if the flow has vertical components, then the associated nontrivial convective term competes with the nonlinear degenerate diffusion. Depending on the rock type and initial and boundary conditions of the flow, the solutions may be smooth or may exhibit sharp fronts [4, 5, 6].

The presence of dynamic capillary pressure terms changes the type of original nonlinear degenerate parabolic PDEs to *pseudo-parabolic*, with the additional nonlinear degenerate term being proportional to a coefficient τ, see development in Section 2.3. Available existence, uniqueness, and regularity theory for pseudo-parabolic equations [58, 60, 61] predicts that the additional pseudo-parabolic term decreases the smoothing property characteristic to parabolic problems (if at all present) to a factor involving $e^{-\tau}$. In addition it is known that there is in general no maximum principle for pseudo-parabolic equations such as one expected of solutions to parabolic equations. Finally, we note that the available theory may or may not include cases with dominating and degenerate convection; assumptions need to be verified on a case by case basis.

Numerical methods for Richards' equation include practical implementations of finite difference [69], finite elements [69, 40, 44, 54], finite volumes [1] and characteristic-based methods [7]; we do not attempt to give a comprehensive review here. Typically, convergence results are formulated for either transformed variables or for cases away from degeneracy, or for regularized problem. See, e.g., [9, 36] for results and references using Kirchoff transformation, and [33] for those using similarity solutions. Numerical methods for Richards' equation in physical variables have been used in hydrology [20] but have not been analyzed outside smooth regimes where standard convergence rates apply. Furthermore, there exist a plethora of methods applicable to two-phase flow formulation of unsaturated flow, see [25, 43, 22, 23, 21]; however, similarly to the case of Richards' equation, those results have been formulated for transformed variables or for smooth regime(s). Finally, comparison of two-phase flow versus Richards' formulations have been studied