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BROWNIAN MOTION AND ENTROPY GROWTH ON IRREGULAR SURFACES

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Abstract. Many situations of physical and biological interest involve diffusions on manifolds. It is usually assumed that irregularities in the geometry of these manifolds do not influence diffusions. The validity of this assumption is put to the test by studying Brownian motions on nearly flat 2D surfaces. It is found by perturbative calculations that irregularities in the geometry have a cumulative and drastic influence on diffusions, and that this influence typically grows exponentially with time. The corresponding characteristic times are computed and discussed. Conditional entropies and their growth rates are considered too.

Key Words. Brownian motion, stochastic processes on manifolds, lateral diffusions.

1. Introduction

Stochastic process theory is one of the most popular tools used in modelling time-asymmetric phenomena, with applications as diverse as economics ([21, 22]), traffic management ([20, 15]), biology ([16, 2, 10, 8]), physics ([23]) and cosmology ([5]). Many diffusions of biological interest, for example the lateral diffusions ([4, 17]), can be modelled by stochastic processes defined on differential manifolds ([12, 13, 9, 18]). In practice, the geometry of the manifold is never known with infinite precision, and it is common to ascribe to the manifold an approximate, mean geometry and to assume irregularities in the geometry have, in the mean, a negligible influence on diffusion phenomena ([4, 1, 3, 6, 19]). The aim of this article is to investigate if this last assumption is indeed warranted.

To this end, we fix a base manifold $\mathcal{M}$ and focus on Brownian motion. We introduce two metrics on $\mathcal{M}$. The first one, g, represents the real, irregular geometry of the manifold; what an observer would consider as the approximate, mean geometry is represented by another metric, which we call $\bar{g}$; to keep the discussion as general as possible, both metrics are allowed to depend on time.

We compare the Brownian motions in the approximate metric $\bar{g}$ to those in the real, irregular metric g by comparing their respective densities with respect to a reference volume measure, conveniently chosen as the volume measure associated to $\bar{g}$. Explicit computations are presented for diffusions on nearly flat 2D surfaces whose geometry fluctuates on spatial scales much smaller than the scales on which these diffusions are observed. We investigate in particular if the densities generated by Brownian motions in the real, irregular metric g coincide on large scales with the densities generated by Brownian motions in the approximate metric $\bar{g}$. We perform a perturbative calculation and find that, generically, these densities differ, even on large scales, and that the relative differences of their spatial

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Fourier components grow exponentially in time; on a given surface, the characteristic time τ at which the perturbative terms become comparable (in magnitude) to the zeroth order terms depends on the amplitude ε of the irregularities and on the large scale wave vector k at which diffusions are observed; we find that τ generally scales as $-\left(\nu^{-2}\ln(\varepsilon/\nu^{1/2})\right) \times \left(1/\mid K^* \mid^2 \chi\right)$, where χ is the diffusion coefficient and $\nu = \mid k \mid / \mid K^* \mid$, K^* being a typical wave-vector characterizing the metric irregularities. Our general conclusion is that geometry fluctuations have a cumulative effect on Brownian motion and that their influence on diffusions cannot be neglected.

2. Brownian motions on a manifold

2.1. Brownian motion in a time-independent metric. Let $\mathcal{M}$ be a fixed real base manifold of dimension d. Let g be a time-independent metric on $\mathcal{M}$. This metric endows $\mathcal{M}$ with a natural volume measure which will be denoted hereafter by dVol_g. If $\mathcal{C}$ is a chart on $\mathcal{M}$ with coordinates $x = (x^i), i = 1, ..., d$, integrating against dVol_g comes down to integrating against $\sqrt{\det g_{ij}}\, d^d x$, where the g_{ij}'s are the components of g in the coordinate basis associated to $\mathcal{C}$.

There is a canonical definition of a Brownian motions on $\mathcal{M}$ equipped with metric g ([14, 9, 11, 18]). Quite intuitively, these Brownian motions are defined through the diffusion equation obeyed by their densities n with respect to dVol_g. Given an arbitrary positive diffusion constant χ, this equation reads:

$$\partial_t n = \chi \Delta_g n, \tag{1}$$

where Δ_g is the Laplace-Beltrami operator associated to g ([7]); given a chart $\mathcal{C}$ with coordinates x, one can write:

$$\Delta_g n = \frac{1}{\sqrt{\det g_{kl}}} \partial_i \left(\sqrt{\det g_{kl}}\, g^{ij} \partial_j n \right), \tag{2}$$

where ∂_i represents partial derivation with respect to x^i and the g^{ij}'s are the components of the inverse of g in the coordinate basis associated to $\mathcal{C}$. Observe that one of the reasons why this definition makes sense is that the diffusion equation (1) conserves the normalization of n with respect to dVol_g.

2.2. Brownian motion in a time-dependent metric. The preceding definition of Brownian motion cannnot be used in this case because the diffusion equation (1) does not conserve the normalization of $n(t)$ with respect to the volume measure $\mathrm{dVol}_{g(t)}$ associated to a time-dependent metric. To proceed, we introduce an arbitrary, time-independent metric γ on $\mathcal{M}$, denote by $\mu_{g(t)|\gamma}$ the density of $\mathrm{dVol}_{g(t)}$ with respect to dVol_γ, and define the Brownian motion in the time-dependent metric $g(t)$ as the stochastic process whose density n with respect to $\mathrm{dVol}_{g(t)}$ obeys the following generalized diffusion equation:

$$\frac{1}{\mu_{g(t)|\gamma}} \partial_t \left(\mu_{g(t)|\gamma} n \right) = \chi \Delta_{g(t)} n. \tag{3}$$

Given an arbitrary coordinate system (x), equation (3) transcribes into:

$$\partial_t \left(\sqrt{\det g_{kl}}\, n \right) = \chi \partial_i \left(\sqrt{\det g_{kl}}\, g^{ij} \partial_j n \right), \tag{4}$$