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CONVERGENCE OF HIGH ORDER METHODS FOR MISCIBLE DISPLACEMENT

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Abstract. We derive error estimates for a fully discrete scheme using primal discontinuous Galerkin discretization in space and backward Euler discretization in time. The estimates in the energy norm are optimal with respect to the mesh size and suboptimal with respect to the polynomial degree. The proposed scheme is of high order as polynomial approximations of pressure and concentration can take any degree. In addition, the method can handle different types of boundary conditions and is well-suited for unstructured meshes.

Key Words. flow, transport, porous media, miscible displacement, NIPG, SIPG, IIPG, h and p-version, fully discrete scheme.

1. Introduction

A high order numerical method for solving miscible displacement is introduced and analyzed in this paper. Miscible displacement occurs in important applications such as remediation of contaminated groundwater and production of oil from petroleum reservoirs. The physical model that describes the miscible displacement phenomena arises from the natural law of conservation of mass. This law is applied to each component of the fluid mixture. The mathematical model consists of a coupled system of partial differential equations: a pressure equation and a concentration equation for each component. Since the components of the fluid mixture may react with each other, the numerical method must accurately solve the laws of conservation. In particular, it is important to solve the continuity equation that describes the flow phenomena with high accuracy.

In this work, we propose a fully discrete scheme that is locally mass conservative. The approximations of pressure and concentration at each time step are discontinuous piecewise polynomials of different degrees. We show convergence of the numerical method with respect to both the mesh size and the polynomial degree. The flexibility inherent to discontinuous approximation spaces allows the use of complicated geometries and unstructured meshes. The primal discontinuous Galerkin method, analyzed in this paper, encompasses the nonsymmetric interior penalty Galerkin (NIPG) method, the symmetric interior penalty Galerkin (SIPG) and the incomplete interior penalty Galerkin (IIPG) method introduced for elliptic problems in [18, 26, 4]. Discontinuous Galerkin methods have been recently popular in modeling complex flow and transport problems in porous media (see for instance [22, 6, 5, 10, 14]).

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Several methods for solving the miscible displacement are proposed and analyzed in the literature. When classical continuous finite element approximations are used for both the pressure and the concentration equations, optimal convergence rates are proved in the dispersion-free case and nearly optimal convergence rates in the dispersion case, under somewhat idealized circumstances [8]. However, this procedure does not handle the transport-dominated problem arising from the concentration equation. Strong improvement in the accuracy of the approximation of the concentration is obtained by considering interior penalty Galerkin methods that can be based on continuous piecewise polynomial spaces [27] or on discontinuous piecewise polynomial spaces [11]. In this case, the pressure equation is solved with a continuous finite element method and penalty terms involving the jumps in the normal derivative are introduced in the concentration equation.

In the miscible displacement problem, only the velocity enters the equation for the concentration and therefore a natural procedure for solving the pressure equation is the locally mass conservative mixed finite element method. The concentration equation can be handled either by a continuous finite element method [12, 13] or by a modified method of characteristics, which combines the time derivative and the advection terms as a directional derivative [16, 24, 3]. In [23], a combination of a continuous finite element method and the method of characteristics for the concentration equation and a standard continuous finite element method for the pressure equation is used. As in the above cases, time stepping is done along the characteristics.

More recently, primal discontinuous Galerkin methods have been applied and analyzed for solving the miscible displacement problem using a semi-discrete approach. The system of equations is discretized in space only. A combined mixed method for the pressure equation with NIPG for concentration equation is studied in [20]. Both pressure and concentration are approximated by the NIPG method in [21, 17]. However, the convergence result in [21] is valid only if the boundary condition for pressure is a Neumann type. The numerical scheme presented in this paper, is fully discrete and valid for both Dirichlet and Neumann boundary conditions for the pressure and Dirichlet, Neuman and mixed boundary conditions for the concentration.

The outline of the paper is as follows. Section 2 contains the model problem and assumptions on the data. The coupled discontinuous Galerkin scheme is formulated in Section 3. Existence and convergence of the numerical solution are obtained in Section 4. Extensions of the scheme to several types of boundary conditions are presented in Section 5.

2. Model Problem and Notation

Consider the miscible displacement of one incompressible fluid by another in a porous medium $\Omega \subset \mathbb{R}^2$ and over the time interval $(0,T)$. Let p denote the pressure in the fluid mixture and let c denote the concentration (fraction volume) of the displaced fluid in the fluid mixture. The partial differential equations describing