INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Supp , Pages 64–84

WAVELETS, A NUMERICAL TOOL FOR MULTISCALE PHENOMENA: FROM TWO DIMENSIONAL TURBULENCE TO ATMOSPHERIC DATA ANALYSIS.

PATRICK FISCHER AND KA-KIT TUNG

Abstract. Multiresolution methods, such as the wavelet decompositions, are increasingly used in physical applications where multiscale phenomena occur. We present in this paper two applications illustrating two different aspects of the wavelet theory.
In the first part of this paper, we recall the bases of the wavelets theory. We describe how to use the continuous wavelet decomposition for analyzing multifractal patterns. We also summarize some results about orthogonal wavelets and wavelet packets decompositions.
In the second part, we show that the wavelet packet filtering can be successfully used for analyzing two-dimensional turbulent flows. This technique allows the separation of two structures: the solid rotation part of the vortices and the remaining mainly composed of vorticity filaments. These two structures are multiscale and cannot be obtained through usual filtering methods like Fourier decompositions. The first structures are responsible for the inverse transfer of energy while the second ones are responsible for the forward transfer of enstrophy. This decomposition is performed on numerical simulations of a two dimensional channel in which an array of cylinders perturb the flow.
In the third part, we use a wavelet-based multifractal approach to describe qualitatively and quantitatively the complex temporal patterns of atmospheric data. Time series of geopotential height are used in this study. The results obtained for the stratosphere and the troposphere show that the series display two different multifractal behaviors. For large time scales (several years), the main Hölder exponent for the stratosphere and the troposphere data are negative indicating the absence of correlation. For short time scales (from few days to one year), the stratopshere series present some correlations with Hölder exponents larger than 0.5, whereas the troposhere data are much less correlated.

Key Words. Wavelets, two dimensional turbulence, multifractal analysis, atmospheric data

1. Review on wavelets

The one dimensional wavelet theory is reviewed in this part. The generalization to higher dimension is relatively easy and is based on tensor products of one dimensional basis functions. The two dimensional wavelet theory is recalled here in the wavelet packets framework only. We present here a summary of the theory, and a more complete description can be found in [12, 26].
Any time series, which can be seen as a one dimensional mathematical function, can

2000 *Mathematics Subject Classification.* 65T60, 76F65, 28A80.

be represented by a sum of fundamental or simple functions called basis functions. The most famous example, the Fourier series,

$$s(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ikt} \tag{1}$$

is valid for any 2π-periodic function sufficiently smooth. Each basis function, e^{ikt} is indexed by a parameter k which is related to a frequency. In (1), $s(t)$ is written as a superposition of harmonic modes with frequencies k. The coefficients c_n are given by the integral

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} s(t) e^{-ikt} dt \tag{2}$$

Each coefficient c_k can be viewed as the average harmonic content of $s(t)$ at frequency k. Thus the Fourier decomposition gives a frequency representation of any signal. The computation of c_k is called the decomposition of s and the series on the right hand side of (1) is called the reconstruction of s.

Although this decomposition leads to good results in many cases, some disadvantages are inherent to the method. One of them is the fact that all the information concerning the time variation of the signal is completely lost in the Fourier description. For instance, a discontinuity or a localised high variation of the frequency will not be well described by the Fourier representation. The underlying reason lies in the nature of complex exponential functions used as basis functions. They all cover the entire real line, and differ only with respect to frequency. They are not suitable for representing the behaviour of a discontinuous function or a signal with high localised oscillations.

Like the complex exponential functions of the Fourier decomposition, wavelets can be used as basis functions for the representation of a signal. But, unlike the complex exponential functions, they are able to restore the positional information as well as the frequency information.

1.1. Continuous wavelets and the multifractal formalism. The wavelet-based multifractal formalism has been introduced in the nineties by Mallat [25, 26], Arneodo [2, 3, 4], Bacry [5] and Muzy [28]. A wavelet transform can focus on localized signal structures with a zooming procedure that progressively reduces the scale parameter. Singularities and irregular structures often correspond to essential information in a signal. The local signal regularity can be described by the decay of the wavelet transform amplitude across scales. Singularities can be detected by following the wavelet transform local maxima at fine scales.

The wavelet transform is a convolution product of a data sequence with the compressed (or dilated) and translated version of a basis function ψ called the wavelet mother. The scaling and translation are performed by two parameters: the scale parameter a dilates or compresses the mother wavelet to various resolutions and the translation parameter b moves the wavelet all along the sequence:

$$WT_s(b,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} s(t) \psi^* \left(\frac{t-b}{a} \right) dt, \; a \in \mathbb{R}^{+*}, b \in \mathbb{R}. \tag{3}$$

This definition of the wavelet transform leads to an invariant L^2 measure, and thus conserves the energy ($\|s\|_2 = \|WT_s\|_2$). A different normalization could be used leading to a different invariant.