INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 1, Pages 161–176

NUMERICAL MODELING OF A DUAL VARIATIONAL INEQUALITY OF UNILATERAL CONTACT PROBLEMS USING THE MIXED FINITE ELEMENT METHOD

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Abstract. We study the dual mixed finite element approximation of unilateral contact problems. Based on the dual mixed variational formulation with three unknowns (stress, displacement and the displacement on the contact boundary), the a priori error estimates have been established for both conforming and nonconforming finite element approximations. A Uzawa type iterative algorithm is developed to solve the resulting linear system. Numerical example shows good performance of the method.

Key Words. mixed finite element method, dual variational inequality, Uzawa algorithm and error estimates.

1. Introduction

While contact problems are being solved and many finite element programs offer contact analysis capabilities for production and research, efforts to obtain more effective solutions are still made (cf. [2]). One reason is that many different kinds of contact problems can involve large relative motion, frictional forces, and static or dynamic condition. Another reason is that contact solution procedures only stay in research and easy-to-use finite element schemes for contact problems are still unavailable in applications.

Developing efficient computing tools for the numerical simulation of contact problem with unilateral Signorini boundary conditions is of a permanent growing interest in many physical areas (cf. [2, 3, 4, 15, 16, 18]). The particular feature of the unilateral problems is that the mathematical variational statement leads to variational inequalities set on closed convex functional cones. The modeling of the non-penetration condition in the discrete finite element level is of crucial importance. This condition may be imposed on the displacement and expressed in a weak sense. The way that enforced depends on the well-posedness of the discrete inequalities and the accuracy of the approximation algorithm. This point is addressed in many published papers, especially for Lagrangian finite element discretizations (cf. [18, 16, 17, 3, 19]). In these papers, either the displacement is the only unknown or the displacement and the stress on the contact zone are independent unknowns. The convergence rate of these methods have been established. Much attention has been paid to the numerical simulation of variational inequalities modeling for unilateral contact problems by finite element methods (cf. [3, 9, 11, 16]). Either from the accuracy point of view or from developing efficient algorithms to solve the resulting minimization problem(cf. [20]), the hardest task is the discretization of

Received by the editors May 10, 2007 and, in revised form, May 18, 2008.
2000 *Mathematics Subject Classification.* 35R35, 49J40 , 60G40.
The project was supported by the National Natural Sciences Foundation of China (No.40745033) and by the Natural Science Foundation of Hebei Province (No.A2007001027) .

the Signorini unilateral condition, which usually can not be satisfied *exactly* by the numerical solution. This often leads to nonconforming method (cf. [19]).

When high accuracy of the stress and the displacement on the contact boundary are desirable, a possible way is to successively refine the mesh. An alternative way is to resort to a new mixed variational formulation that includes stress, displacement and the displacement on the contact boundary as the main unknowns. However, the well-posedness of such mixed variational problem depends heavily on the so-called ellipticity condition and the B.-B. inequality, which is the source of trouble in constructing finite element approximation spaces. To overcome this difficulty, we modify the dual variational formulation by adding a new term that enhances the ellipticity and brings more freedom in choosing finite element spaces. Based on this modified variational formulation problem, we introduce two new types of finite element approximation spaces. A priori error estimates have been carried out for both methods and our numerical results confirm the efficiency of the methods.

The remaining part of this paper is as follows. In the next section we will state the functional setting and the unilateral contact problem. In Section 3, we shall derive the dual variational formulation of the unilateral contact problems by Lagrange multiplier method. In Section 4, we introduce the conforming and nonconforming finite element approximations of the dual mixed variational problem and derive the a priori error estimates. In Section 5, a Uzawa type iterative algorithm is presented to solve the approximation problem. Numerical study of two methods is described in Section 6. Finally, we give concluding remarks in Section 7.

2. Functional setting and contact Problem

Notation. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain with generic point x. The Lebesgue space $L^p(\Omega)$ is endowed with the norm: $\forall \psi \in L^p(\Omega)$,

$$\|\psi\|_{L^p(\psi)} = \left(\int_\Omega |\psi(x)|^p dx \right)^{1/p}.$$

We make use of the standard Sobolev space $H^m(\Omega),\ m \geq 1$, equipped with the norm:

$$\|\psi\|_{H^m(\Omega)} = \left(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha \psi(x)\|^2_{L^2(\Omega)} \right)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index in N and the symbol ∂^α represents a partial derivative. In particular, $L^2(\Omega) = H^0(\Omega)$. The fractional order Sobolev space $H^\nu(\Omega),\ \nu \in \mathbb{R}_+ \setminus N$ is defined as in [1] and equipped with the norm

$$\|\psi\|_{H^\nu(\Omega)} = \left(\|\psi\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{(\partial^\alpha \psi(x) - \partial^\alpha \psi(y))^2}{|x-y|^{2+2\theta}} dxdy \right)^{1/2},$$

where $\nu = m + \theta,\ m$ is the integer part of ν and $\theta \in [0,1]$ is the decimal part.

For any portion γ of the boundary $\partial\Omega$ and any $\nu \in R_+ \setminus N$, the Hilbert space $H^\nu(\gamma)$ is associated with the norm

$$\|\psi\|_{H^\nu(\gamma)} = \left(\|\psi\|^2_{H^m(\gamma)} + \int_\gamma \int_\gamma \frac{(\partial^m_\Gamma \psi(x) - \partial^m_\Gamma \psi(y))^2}{|x-y|^{2+2\theta}} d\Gamma d\Gamma \right)^{1/2},$$

where m is the integer part of ν, θ its decimal part. The symbol $\partial^m_\Gamma \psi$ stands for the m-th order derivative of ψ along γ and $d\Gamma$ denotes the linear measure on $\partial\Omega$. The space $H^{-\nu}(\gamma)$ stands for the topological dual space of $H^\nu(\gamma)$ and the duality pairing is denoted $< .,. >_{\nu,\gamma}$. The spacial space $H^{m+\frac{1}{2}}_{00}(\gamma)$ is defined as the set