INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 1, Pages 17–32

ADAPTIVE FINITE ELEMENT METHODS FOR PARAMETER ESTIMATION PROBLEMS IN LINEAR ELASTICITY

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Abstract. In this paper, the Lamé coefficients in the linear elasticity problem are estimated by using the measurements of displacement. Some a posteriori error estimators for the approximation error of the parameters are derived, and then adaptive finite element schemes are developed for the discretization of the parameter estimation problem, based on the error estimators. The Gauss-Newton method is employed to solve the discretized nonlinear least-squares problem. Some numerical results are presented.

Key Words. parameter estimation, finite element approximation, adaptive finite element methods, a posteriori error estimates, linear elasticity.

1. Introduction

In this paper, we consider a parameter identification problem in the linear elasticity problem

$$\begin{aligned} -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) &= \mathbf{f} \qquad in \ \ \Omega, \\ \mathbf{u} &= \mathbf{u}_D \qquad on \ \ \Gamma_D \end{aligned} \tag{1}$$

where Ω is a polygonal domain in two dimensional space with the Lipschitz-continuous boundary, and the boundary Γ_D is positive $d\gamma-$measurable. As usual, $\mathbf{u}$ denotes the displacement, and $\mathbf{f}$ and $\mathbf{u}_D$ represent the body force and the boundary displacement, respectively. Let $U = \left\{\mathbf{u} \in (H_0^1(\Omega))^2\right\}$ and $\mathbf{u} \in U + \mathbf{u}_D = Y$, where the space U is assumed to have the product norm

$$\mathbf{u} = (u_1, u_2) \to ||\mathbf{u}||_{1,\Omega} = (\sum_{i=1}^{2} ||u_i||_{1,\Omega}^2)^{\frac{1}{2}}.$$

We define the strain tensor $(\epsilon_{ij}(\mathbf{u}))$ as

$$\epsilon_{ij}(\mathbf{u}) = \epsilon_{ji}(\mathbf{u}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad 1 \le i,j \le 2,$$

and the stress tensor (σ_{ij}) is then given by Hooke's law for isotropic bodies

$$\sigma_{ij}(\mathbf{u}) = \sigma_{ji}(\mathbf{u}) = \lambda(\sum_{k=1}^{2} \epsilon_{kk}(\mathbf{u}))\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u}), \quad 1 \le i,j \le 2,$$

where δ_{ij} is Kronecker's symbol. The Lamé coefficients λ and μ are given by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

with Poisson's ratio ν and Young's modulus E. It is well known that $\lambda \ge \lambda' > 0$ and $\mu \ge \mu' > 0$.

Received by the editors November 28, 2007 and, in revised form, December 22, 2007.
2000 *Mathematics Subject Classification.* 65N30, 49J20, 74B05, 65M32.

In our parameter estimation problem, we aim to recover the constants λ and μ by using the known measurements of displacement $\mathbf{u}$. To this end, the well-known output least-squares formulation is used, i.e., we solve

$$\min_{\mathbf{m}} \frac{1}{2}||Q\mathbf{u}(\mathbf{m}) - \mathbf{z}||_Z^2, \tag{2}$$

where $\mathbf{u}$ is the solution of the linear elastic equation (1) and $\mathbf{m} = (m_1, m_2)^T = (\lambda, \mu)^T$. The vector $\mathbf{z} \in Z$ is a given set of measurements and the observation space Z is supposed to be a Hilbert space. Furthermore, we set $Q : Y \to Z$ as a linear bounded observation operator.

Usually, the parameter estimation problem is ill-posed or ill-conditioned; see [16]. Some regularization terms are added to the cost function (2) such that

$$\min_{\mathbf{m}} \left\{ \frac{1}{2}||Q\mathbf{u}(\mathbf{m}) - \mathbf{z}||_Z^2 + \frac{\beta}{2}||\mathbf{m} - \mathbf{m}_{ref}||^2 \right\}, \tag{3}$$

where the penalty parameter β is assumed to be a very small positive number and $\mathbf{m}_{ref}$ is a reference model. The regularization term $\beta/2 \cdot ||\mathbf{m} - \mathbf{m}_{ref}||^2$ improves the conditioning of the inverse problem. Here $||\cdot||$ denotes the l^2 norm of the vector. A good regularization parameter β should yield a fair balance between the perturbation error and the regularization error. Assume that the data $\mathbf{z}$ contains noise with known standard deviation $\mathbf{e}$, then the regularization parameter should be chosen such that

$$||Q\mathbf{u}(\mathbf{m}) - \mathbf{z}||_Z = ||\mathbf{e}||_Z;$$

see [23]. To solve the problems without known deviation, methods such as L-curve criterion, generalized cross-validation and the quasi-optimality criterion can be used for the regularization parameter selection; for more details, see [15, 23, 29].

To solve the parameter estimation problem, one must approximate the infinite-dimensional problem by introducing discretizations for the state space Y such as a finite element or difference approach. It is clear that the efficiency of our numerical methods will be influenced by the discretization scheme. In recent years, adaptive finite element method has been extensively and successfully investigated; see [1]. By using the adaptive finite element method, a numerical solution with a prescribed tolerance can be obtained with a minimal amount of work. This ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate. Although adaptive finite element approximation is widely used in the numerical simulation, it is not yet fully utilized in the parameter estimation problem. Very recently, some a posteriori error estimators have been derived for the parameter estimation problem [4, 8, 19]. In this paper, an adaptive finite element method for our parameter estimation problems is developed. Our emphasis here is to derive some a posteriori error estimators which control the error in the unknown parameters, instead of the cost function [4, 7]. Moreover, these error estimators are used to guide our mesh refinement.

We note that some efficient a posteriori error estimators have been derived by using the adjoint equation approach [4, 8]. In these error estimators, the local residuals of the solution are multiplied by weights which measure the dependence of the error on the local residuals. The weights are obtained by approximately solving an adjoint problem. However, the exact solution itself is included in these error estimators, which must be approximated by techniques such as higher order interpolation. Furthermore, since $\lambda \geq \lambda' > 0$ and $\mu \geq \mu' > 0$, we get inequality constraints (see the optimal conditions (8) in section 2). In general, it is not clear how to apply the adjoint approach to this inequality constraint minimization problem. Thus, our corresponding error estimators are based on the approach developed by Kunisch, Liu and Yan [19]. In our error estimator, the weights are absorbed to