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DISCRETE MAXIMUM PRINCIPLES FOR FEM SOLUTIONS OF SOME NONLINEAR ELLIPTIC INTERFACE PROBLEMS

JÁNOS KARÁTSON AND SERGEY KOROTOV

Abstract. Discrete maximum principles are proved for finite element discretizations of nonlinear elliptic interface problems with jumps of the normal derivatives. The geometric conditions in the case of simplicial meshes are suitable acuteness or nonobtuseness properties.

Key Words. Nonlinear elliptic problem, interface problem, maximum principle, discrete maximum principle, finite element method, simplicial mesh.

1. Introduction

The maximum principle forms an important qualitative property of second order elliptic boundary value problems [12, 25, 29]. Consequently, the discrete analogues of the maximum principle (so-called discrete maximum principles, DMPs) have drawn much attention. Various DMPs have been formulated and proved including the case of finite difference, finite volume and finite element approximations, and corresponding geometric conditions on the computational meshes have been given, see, e.g., [3, 5, 6, 7, 9, 13, 21, 30, 31, 33] for linear and [16, 17, 22] for nonlinear problems with standard (i.e., Dirichlet, and in [16, 17] mixed) boundary conditions.

In this paper we address interface problems, which arise in various branches of material science, biochemistry, multiphase flow etc., often when two or more distinct materials are involved with different conductivities or densities. Another (for our work, motivating) example is from localized reaction-diffusion problems [14, 15], see at the end of this paper. Many special numerical methods have been designed for interface problems, see, e.g., [14, 27, 28, 26], but maximum principles have received less attention than for the case of standard boundary value problems. A continuous minimum principle for a related problem is given in [11]. The discrete maximum principle for suitable finite difference discretizations of linear interface problems has been proved in [27].

Our goal is to derive maximum principles for nonlinear elliptic interface problems when finite element discretization is involved. The present paper is the extension of our paper [16] to a class of such problems, and relies on a similar technique using weak formulation and positivity conditions that ensure well-posedness. Our result is based on the observation that we can recast the considered interface problem to a weak formulation, which is similar to that of the mixed problem studied in [16]. We consider matching conditions for the solution on the interface, i.e., the jump is allowed for the normal derivatives but not for the solution itself. Problems

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with jump of the solution or without well-posedness may be the subject of further research.

The paper is organized as follows. The formulation of the problem is presented in Section 2 with focus on the suitable weak form of the problem, and a corresponding continuous maximum principle is enclosed. The finite element discretization is described in Section 3. Discrete maximum principles are derived and examples are given in Section 4.

2. Nonlinear elliptic interface problems

2.1. Formulation of the problem. We investigate nonlinear interface problems of the following form:

$$\begin{cases} -\operatorname{div}\Big(b(x,\nabla u)\,\nabla u\Big) + q(x,u) \;=\; f(x) \quad \text{in } \Omega\setminus\Gamma, \\ [\,u]_\Gamma \;=\; 0 \qquad \text{on } \Gamma, \\ \big[\,b(x,\nabla u)\frac{\partial u}{\partial\nu}\big]_\Gamma + s(x,u) \;=\; \gamma(x) \quad \text{on } \Gamma, \\ u \;=\; g(x) \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\partial\Omega$ denotes the boundary of the domain Ω and the interface Γ is a surface lying in Ω, further, ν denotes the outward normal unit vector, $[\,u]_\Gamma$ and $\big[\,b(x,\nabla u)\frac{\partial u}{\partial\nu}\big]_\Gamma$ denote the jump (i.e., the difference of the limits from the two sides of the interface Γ) of the solution u and the flux $b(x,\nabla u)\frac{\partial u}{\partial\nu}$, respectively. We impose the following

Assumptions 2.1:

(A1) Ω is a bounded open domain in $\mathbf{R}^d$, $d\in\{1,2,\dots\}$ the interface $\Gamma\subset\Omega$ and the boundary $\partial\Omega$ are piecewise smooth and Lipschitz continuous $(d-1)$-dimensional surfaces.

(A2) The scalar functions $b:\Omega\times\mathbf{R}^d\to\mathbf{R}$, $q:\Omega\times\mathbf{R}\to\mathbf{R}$ and $s:\Gamma\times\mathbf{R}\to\mathbf{R}$ are measurable and bounded w.r.t. their first variable $x\in\Omega$ (resp. $x\in\Gamma$) and continuously differentiable w.r.t. their second variable $\eta\in\mathbf{R}^d$ (resp. $\xi\in\mathbf{R}$). Further, $f\in L^2(\Omega)$, $\gamma\in L^2(\Gamma)$ and $g\in H^1(\Omega)$.

(A3) The function b satisfies

$$0<\mu_0\le b(x,\eta)\le\mu_1 \tag{2}$$

with positive constants μ_0 and μ_1 independent of (x,η), further, the diadic product matrix $\eta\cdot\frac{\partial b(x,\eta)}{\partial\eta}$ is symmetric positive semidefinite and bounded in matrix norm by some positive constant μ_2 independent of (x,η).

(A4) Let $2\le p_1$ if $d=2$, or $2\le p_1\le\frac{2d}{d-2}$ if $d>2$, further, let $2\le p_2$ if $d=2$, or $2\le p_2\le\frac{2d-2}{d-2}$ if $d>2$. There exist functions $\alpha_1\in L^{d/2}(\Omega)$, $\alpha_2\in L^{d-1}(\Gamma)$ and a constant $\beta\ge 0$ such that for any $x\in\Omega$ (or $x\in\Gamma$, resp.) and $\xi\in\mathbf{R}$

$$0\le\frac{\partial q(x,\xi)}{\partial\xi}\le\alpha_1(x)+\beta|\xi|^{p_1-2},\qquad 0\le\frac{\partial s(x,\xi)}{\partial\xi}\le\alpha_2(x)+\beta|\xi|^{p_2-2}. \tag{3}$$

Remark 2.1. Problem (1) contains some widespread interface models as special cases, see, e.g., [15, 28] and also the models addressed in subsection 4.4.

Remark 2.2. (i) The role of assumption (A3) is to ensure that the Jacobian matrices $J(x,\eta):=\frac{\partial}{\partial\eta}\Big(b(x,\eta)\,\eta\Big)$ are symmetric and satisfy the uniform ellipticity property $\mu_0|\zeta|^2\le\zeta^T J(x,\eta)\,\zeta\le\mu_3|\zeta|^2$, $\quad\zeta\in\mathbf{R}^d$ (with $\mu_3=\mu_1+\mu_2$), which will