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LOCAL ANALYSIS OF THE LOCAL DISCONTINUOUS GALERKIN METHOD WITH THE GENERALIZED ALTERNATING NUMERICAL FLUX FOR TWO-DIMENSIONAL SINGULARLY PERTURBED PROBLEM

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Abstract. In this paper, we analyze the local discontinuous Galerkin method with the generalized alternating numerical flux for two-dimensional singularly perturbed problem with outflow boundary layers. By virtue of the two-dimensional generalized Gauss-Radau projection and energy technique with suitable weight function, we obtain the double-optimal error estimate, namely, the convergence rate in L^2-norm out of the outflow boundary layer is optimal, and the width of boundary layer is quasi-optimal, when piecewise tensor product polynomial space on quasi-uniform Cartesian meshes are used. Numerical experiments are given to verify the theoretical results.

Key words. Local analysis, local discontinuous Galerkin method, generalized alternating numerical flux, error estimate, singularly perturbed problem.

1. Introduction

Let $\Omega = (0, 1)^2$ be the unit square with boundary Γ, and $T > 0$ is a final time. Consider the following two-dimensional singularly perturbed (SP) problem

$$u_t - \varepsilon\triangle u + \boldsymbol{\beta}\cdot\nabla u + cu = f \quad \text{in} \quad \Omega\times(0,T], \tag{1a}$$

with the Dirichlet boundary condition

$$u(x, y, t) = g(x, y, t) \quad \text{on} \quad \Gamma\times(0,T], \tag{1b}$$

and the initial condition

$$u(x, y, 0) = u_0(x, y) \quad \text{in } \Omega. \tag{1c}$$

Here $0 < \varepsilon \ll 1$ is the diffusion coefficient, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ is the convective velocity field. Without loss of generality, we assume β_1 and β_2 are positive constants. We also assume the given functions c, f, g and u_0 are smooth enough.

It is well known that the exact solution of the SP problem (1) may change rapidly in a narrow region nearby the outflow boundaries $x = 1$ and $y = 1$, and it always appear boundary layer with width $O(\varepsilon\log(1/\varepsilon))$. To give a nice numerical result to this problem, many algorithms have been presented and developed [17]. The local discontinuous Galerkin (LDG) method is a special class of DG methods which has received increasing interest during the last two decades. It was firstly introduced by Cockburn and Shu [8] for the convection-diffusion problems, motivated by the successful numerical experiment of Bassi and Rebay [1] for compressible Navier-Stokes problems. Since the discontinuous finite element spaces do not require any continuity at interface boundaries, the LDG method is very good at solving those fast-varying, even those discontinuous solutions [11]. For more knowledge about this method, please refer to the review paper [21] and the reference therein.

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There have been many global error analysis of the LDG methods for convection diffusion problems, for example [3,9,15,19,21], where the exact solution is assumed to be smooth enough in the whole domain. However, for SP problems, the exact solutions often have no uniform smoothness in the whole domain, and the corresponding global results become useless. To show the numerical advantage of the LDG method for SP problems, local analysis has been carried out in [5,6,23], where the *double-optimal* local error estimate was obtained. Here *double-optimal* means that the convergence rate in L^2-norm out of the outflow boundary layer is optimal, and the width of boundary layer is quasi-optimal. Numerical methods related to this topic also include the space-time DG method [13], the interior penalty DG method [10], continuous interior penalty method [2] and so on.

It is worthy to point out that, in [5, 6, 23] the double-optimal error estimates were established for purely alternating numerical flux, which means the purely upwind numerical flux for the convection and the purely alternating fluxes for the diffusion. However, this type of flux is often not easy to define for linear equations with varying-coefficients or even nonlinear equations [4]. From the view of practice, the generalized alternating numerical flux (GANF) is used more widely in the LDG method. Recently, motivated by the optimal error estimate of an upwind-biased DG method [16], we studied the LDG method with GANF for linear convection-diffusion problems in [4]. By virtue of the generalized Gauss-Radau (GGR) projection [15, 16], we obtained the optimal L^2-norm error estimate in the whole domain. Furthermore, by establishing the sharp approximation property of the one-dimensional GGR (1-d GGR) projection with the weight function, we also derived in [7] the double-optimal local error estimate for the one-dimensional SP problem with stationary outflow boundary layer.

The objective of this paper is to extend the results of [7] to the two-dimensional SP problems with stationary outflow boundary layers. We will present the local stability and show the double-optimal local error estimate of LDG method with GANF for $\mathcal{Q}^k$ element on quasi-uniform Cartesian meshes, where $\mathcal{Q}^k$ means the space of polynomials of degree at most $k \geq 0$ in each variable.

As an important ingredient in the local analysis, the weight function must be defined carefully. In this paper, we take it as the exponential decay function along each spatial directions. Besides, to achieve the double optimal local error estimate, our main technique is the two-dimensional GGR (2-d GGR) projection. The corresponding properties of 2-d GGR projections with the weight function are not easy to be established. Specifically, there are mainly two issues we have to consider.

(1) One is to obtain the optimal approximation property of 2-d GGR projections with weight function. Since the 2-d GGR projections have much complex expressions under the Dirichlet boundary condition, the direct manipulations based on the matrix analysis as [7] is much involved. The main difficulty is caused by the definition of GGR projection at the corner points. To overcome this difficulty, we will carefully investigate the structures of coefficient matrices and use some properties of tensor product of matrices.
(2) The other is to get the superconvergence property of 2-d GGR projections with weight function. Different from one dimensional case, the approximation errors for 2-d GGR projections can not be completely eliminated, in each element and on the interior element boundaries. To derive the optimal error estimate, we need to explore the superconvergence property, which has been discussed in [4], where the 2-d GGR projection under periodic boundary condition was considered. However, in the local error