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L^2 NORM EQUIVALENT A POSTERIORI ERROR ESTIMATE FOR A CONSTRAINED OPTIMAL CONTROL PROBLEM

LIANG GE, WENBIN LIU, AND DANPING YANG

Abstract. Adaptive finite element approximation for a constrained optimal control problem is studied. A posteriori error estimators equivalent to the L^2 norm of the approximation error are derived both for the state and the control approximation, which are particularly suitable for an adaptive multi-mesh finite element scheme and applications where L^2 error is more important. The error estimators are then implemented and tested with promising numerical results.

Key Words. convex optimal control problem, adaptive finite element method, L^2 norm equivalent a posteriori error estimate, multi-meshes.

1. Introduction

There has been so extensive research on developing adaptive finite element algorithms for PDEs in the scientific literature that it is simply impossible to give even a very brief review here. Recently, there has been intensive research in adaptive finite element methods for optimal control problems, see, for example, [2, 3, 4, 6, 8, 9, 12, 15, 16, 18]. Particularly a posteriori error estimates equivalent to the energy norm of the approximation error were derived for several types of optimal control problems. Furthermore it has been found that for constrained control problems, different adaptive meshes are often needed for the control and the states, see [10]. Using different adaptive meshes for the control and the state allows very coarse meshes to be used in solving the state and co-state equations. Thus much computational work can be saved since one of the major computational loads in computing optimal control is to solve the state and co-state equations repeatedly. This will be also seen from our numerical experiments in Section 4.

Although a posteriori error estimates equivalent to the H^1 norm of the approximation error (to be called H^1 norm equivalent a posteriori error estimates) have been derived for several elliptic optimal control problems, see [8, 9, 10], both for the control constraints of obstacle types and integral types, there seems no existing work on L^2 norm equivalent a posteriori error estimates, which are equivalent to the L^2 norm of the approximation error, although some upper bounds were derived using the L^2 norm for the control constraint of an obstacle type, see [10, 16]. It does not seem a trivial problem whether and how some lower bounds can be derived via the L^2 norm, although it seemed possible to adapt the existing duality techniques to derive upper bounds. In many engineering applications, one cares more about averaging values of the control and the states. In these cases, it seems to be more natural to use the L^2-norm of the approximation error as the stopping criteria in

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computations. Thus L^2 equivalent error indicators seem to be quite useful. Error indicators based on the L^2 norm error bounds tend to produce less over-reinterment in such cases.

The purpose of this article is to investigate indicators that are equivalent to the L^2 norm of the approximation error for a constrained optimal control problem, where the control constraint is of an integral type. This control problem was studied in [8, 9], where H^1 norm equivalent a posteriori error estimates were derived. We derived L^2 norm equivalent a posteriori error estimators, which allow different meshes to be used for the stats and the control. Then we performed some numerical tests to confirm the effectiveness of the error estimators.

The plan of the paper is as follows. In Section 2, we will construct the finite element approximation for the distributed optimal control problem. In Section 3, the a posteriori error estimators equivalent in the L^2 norm are derived for the control problem. Finally numerical test results are presented in Section 4.

2. Optimal control problem and its finite element approximation

Let Ω be a bounded open set in R^d $(1 \leq d \leq 3)$ with the Lipschitz boundary $\partial\Omega$. We adopt the standard notation $W^{1,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{1,q}(\Omega)}$ and seminorm $|\cdot|_{W^{1,q}(\Omega)}$ for $1 \leq q \leq \infty$. We set $W_0^{1,q}(\Omega) \equiv \{w \in W^{1,q}(\Omega) : w|_{\partial\Omega} = 0\}$ and denote $W^{1,2}(\Omega)$ $(W_0^{1,2}(\Omega))$ by $H^1(\Omega)$ $(H_0^1(\Omega))$. In the rest of the paper, we will take the state space $V = H_0^1(\Omega)$ and the control space $U = L^2(\Omega)$. Other cases can be considered similarly. Let the observation space $Y = L^2(\Omega)$. We investigate the following distributed convex optimal control problem:

$$\min_{u\in K} \frac{1}{2}\int_\Omega (y - y_d)^2 + \frac{1}{2}\int_\Omega u^2, \tag{2.1}$$

$$-\Delta y = f + u \quad \text{in } \Omega\,, \quad y|_{\partial\Omega} = 0,$$

where $K = \{v \mid v \in L^2(\Omega),\ \int_\Omega v \geq 0\}$ is a closed convex set. We first give a weak formula for the state equation. Let

$$a(y, w) = \int_\Omega \nabla y \cdot \nabla w, \quad \forall\ y, w \in V$$

and

$$(u, v) = \int_\Omega uv, \quad \forall\ u, v \in L^2(\Omega).$$

It follows that

$$\alpha\|y\|_V^2 \leq a(y, y), \quad |a(y, w)| \leq M\|y\|_V\|w\|_V, \qquad \forall\ y, w \in V, \tag{2.2}$$

where $0 < \alpha \leq M < \infty$ are positive constants. With these notions the standard weak formula for the state equation reads: *find* $y \in V$ *such that*

$$a(y, w) = (f + u, w), \qquad \forall\ w \in H_0^1(\Omega). \tag{2.3}$$

Then the mentioned-above control problem can be restated as follows:

$$(\text{OCP}): \quad \begin{cases} \min\limits_{u\in K} \Big\{\frac{1}{2}\int_\Omega (y - y_d)^2 + \frac{1}{2}\int_\Omega u^2\Big\}, \\ a(y, w) = (f + u, w), \quad \forall\ w \in V. \end{cases} \tag{2.4}$$

It follows from [14] that the control problem (OCP) has a unique solution (y, u). Furthermore a pair (y, u) is the solution of (OCP) iff there is a co-state $p \in V$ such