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MECHANISM OF THE FORMATION OF SINGULARITIES FOR QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract. In this paper the mechanism and the character of the formation of singularities caused by eigenvalues or (and) eigenvectors, respectively, will be discussed for 1-D quasilinear hyperbolic systems.

Key Words. Mechanism, singularities, quasilinear hyperbolic systems.

1. Introduction

We consider the following Cauchy problem for the first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

$$t = 0: \quad u = \varphi(x), \tag{1.2}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x), $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, and $\varphi(x)$ is C^1 vector function of x with bounded C^1 norm.

By strict hyperbolicity, on the domain under consideration $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.3}$$

For $i = 1, \cdots, n$, let $l_i(u) = (l_{1i}(u), \cdots, l_{ni}(u))$ (resp., $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp., right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp., } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.4}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{resp., } \det|r_{ij}(u)| \neq 0), \tag{1.5}$$

and all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$. Without loss of generality, we assume that

$$l_i(u)r_j(u) = \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{1.6}$$

where (δ_{ij}) stands for the Kronecker's symbol.

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Using left eigenvectors $l_i(u)$ $(i = 1, \cdots, n)$, system (1.1) can be equivalently rewritten in the following characteristic form

$$l_i(u)\Big(\frac{\partial u}{\partial t} + \lambda_i(u)\frac{\partial u}{\partial x}\Big) = 0 \quad (i = 1, \cdots, n). \tag{1.7}$$

The i-th equation in (1.7) contains only the directional derivatives of u with respect to t along the i-th characteristic direction $dx/dt = \lambda_i(u)$.

By local existence and uniqueness of C^1 solution to the Cauchy problem (cf. [11]), there exists $\delta > 0$ such that Cauchy problem (1.1)–(1.2) admits a unique C^1 solution $u = u(t, x)$ on $0 \le t \le \delta$; moreover, for a given system (1.1), δ may be chosen to depend only on the C^1 norm of φ:

$$\delta = \delta(\|\varphi\|_1), \tag{1.8}$$

where $\|\varphi\|_1 = \|\varphi\|_0 + \|\varphi'\|_0$ in which $\|\varphi\|_0 = \max\limits_{x\in\mathbb{R}} |\varphi|$ is the C^0 norm of φ and $\varphi' = d\varphi/dx$.

Thus, in order to prove the global existence and uniqueness of C^1 solution to Cauchy problem (1.1)–(1.2), one should establish the following uniform a priori estimate: For any given $T_0 > 0$, if Cauchy problem (1.1)–(1.2) admits a unique C^1 solution $u = u(t, x)$ on $0 \le t \le T$ with $0 < T < T_0$, then

$$\|u(t,\cdot)\|_1 \triangleq \|u(t,\cdot)\|_0 + \|u_x(t,\cdot)\|_0 \le C(T_0), \quad \forall\, 0 \le t \le T, \tag{1.9}$$

where $C(T_0)$ is a positive constant independent of T but possibly depending on T_0.

However, it is well-known (cf. [5, 6]) that, generically speaking, the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1)–(1.2) exists only locally in time and the singularity may occur in a finite time, i.e., there exists $t* > 0$ such that as $t \uparrow t*$,

$$\|u(t,\cdot)\|_1 = \|u(t,\cdot)\|_0 + \|u_x(t,\cdot)\|_0 \text{ becomes unbounded.} \tag{1.10}$$

If the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1)–(1.2) blows up in a finite time, we say that there is a formation of singularities. The problem we would like to study is what is the mechanism and the character of the formation of singularities for quasilinear hyperbolic systems. That is to say, in what follows we don't pay our attention on studying if there is a global C^1 solution or if the C^1 solution blows up in a finite time (This is another business on which there are already many results), we study only the mechanism and the character of the formation of singularities under the hypothesis that the formation of singularities occurs.

Obviously, if all eigenvalues λ_i and all left (resp., right) eigenvectors l_i (resp., r_i) $(i = 1, \cdots, n)$ are independent of u, system (1.1) or (1.7) reduces to a linear hyperbolic system with constant coefficients and then there is no singularity at all. Hence, in order that the singularity occurs, it is necessary to have the dependence of eigenvalues or (and) eigenvectors on u.

2. Singularity caused by eigenvalues

In the special case that all left (resp., right) eigenvectors are independent of u, the singularity (if any!) should be caused only by eigenvalues. By the