INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 4, Pages 533–561

DISCONTINUOUS GALERKIN METHODS FOR CONVECTION-DIFFUSION EQUATIONS FOR VARYING AND VANISHING DIFFUSIVITY

J. PROFT AND B. RIVIÈRE

Abstract. This work formulates and analyzes a new family of discontinuous Galerkin methods for the time-dependent convection-diffusion equation with highly varying diffusion coefficients, that do not require the use of slope limiting techniques. The proposed methods are based on the standard NIPG/SIPG techniques, but use special diffusive numerical fluxes at some important interfaces. The resulting numerical solutions have an L^2 error that is significantly smaller than the error obtained with standard discontinuous Galerkin methods. Theoretical convergence results are also obtained.

Key Words. numerical fluxes, discontinuous Galerkin methods, high and low diffusivity, L^2 error

1. Introduction and Problem Definition

In this work we explore the development and analysis of discontinuous Galerkin methods applied to the solution of linear advection-diffusion equations

$$\partial_t u + \nabla \cdot (\beta u - \epsilon \nabla u) = f, \qquad \text{in} \quad \Omega \times (0,T). \tag{1}$$

Although problems of this type occur in many applications, we are primarily motivated by the modeling of flow in porous media such as petroleum reservoir and groundwater aquifer simulation. The physical, geological, and chemical properties of the medium may lead to a degeneracy in the spatially varying diffusion coefficient of the mathematical equations describing the model.

Classical numerical methods exhibit instability in the solution even in the non-degenerate case, when the diffusion coefficient is sufficiently small compared to the advection coefficient. In such a situation, the ratio of advection to diffusion is sufficiently high to impose hyperbolic-type behavior in the solution and the numerical solution is incapable of capturing the resulting boundary layer phenomenon. Consequently, even though sufficient regularity exists in the mathematical description of the problem to expect stable results, the numerical scheme is unable to recognize the existence of small and possibly zero diffusion leading to extreme numerical instabilities. Although this phenomena may be resolved by refinement of the mesh, there is a corresponding considerable increase in computational effort.

Advection-diffusion equations of this type have been discretized using classical finite element and finite difference methods that typically utilize an operator splitting technique to handle the difficulties associated with advective transport and

Received by the editors July 2, 2008 and, in revised form, February 9, 2009.
The second author is grateful for the NSF support through the grants NSF DMS 0506039 and NSF DMS 0810422.

diffusion separately [23, 16]. Such computational methods often utilize slope limiting procedures to supress the amount of unphysical oscillations in the numerical solution or the inclusion of a streamline-diffusion stabilization term [15]. Additionally, domain decomposition techniques utilizing differing numerical methods on distinct subdomains have been proposed to model the multi-physics aspects of the problem [12, 25]. In this paper, we propose an adaptive flux technique to maintain stability, based on a discontinuous Galerkin (DG) discretization, that minimizes the L^2 norm of the error and makes the use of slope limiting techniques superfluous.

DG methods possess several characteristics which render them useful in many applications. The flexibility of the method allows for element-wise polynomial degree approximation and general non-conforming meshes. Some well known versions applied to elliptic equations include the symmetric interior penalty method (SIPG) [2], the OBB method [3], the non-symmetric interior penalty Galerkin method (NIPG) [20] and the incomplete interior penalty version (IIPG) [8]. In [14], the analysis is extended to advection-diffusion-reaction problems with variable tensor-valued diffusion but the proposed technique still exhibits the same instabilities as the classically defined DG methods we consider herein. DG methods have been applied to transport equations [19, 24] where the estimates derived are semi-discrete and present numerical examples for constant diffusion only. Alternative DG methods based on the discretization of hyperbolic equations include the local discontinous Galerkin method [6], subsequently extended by various authors to advection-diffusion equations.

The case of a spatially dependent, possibly degenerate diffusion coefficient has not been analyzed previously in the context of DG methods. In this work, our focus is to improve the numerical results in the case of a small (and possibly degenerate) diffusion coefficient without resorting to the use of slope-limiters nor the considerable increase in computational cost associated with mesh refinement. Under the assumption that the mesh fits the discontinuities of the diffusion coefficient, our scheme successfully detects the difficult boundary layer region and adaptively switches techniques to maintain stability. The boundary layer region occurs when the advection-diffusion ratio is sufficiently high that the method cannot resolve the small scale solution behavior. Instead, it treats the problem as the degenerate diffusion case where sufficient mathematical regularity does not exist to justify use of the SIPG/NIPG method. Indeed, the use of an averaged flux is only valid in the case of a continuous solution, which is not mathematically accurate in the degenerate diffusion case at the interface from low to high diffusivity. Only when the advection-diffusion ratio is relatively small can the original numerical technique recognize the small scale phenomena, i.e. non-degenerate diffusion. Our adaptive method automatically recognizes these regions of numerical instability and successfully produces an accurate, stable, and relatively efficient solution.

Verification is the process of demonstrating that a computational model accurately approximates the exact solution to a mathematical model. The identification and quantification of errors in the corresponding numerical implementation is a central component of this process [1, 22]. Our paper deals with verification in the sense that we show that the standard DG methods yield poor L^2 accurary with respect to benchmark solutions. We propose new adaptive DG methods that solve the mathematical model problem accurately. Moreover, our verification analysis is valid for a spatially varying diffusion coefficient that may possibly be degenerate. We obtain theoretical estimates for the L^2 norm of the error and we show numerically that our proposed method yields a smaller error.