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DYNAMICS AND VARIATIONAL INTEGRATORS OF STOCHASTIC HAMILTONIAN SYSTEMS

LIJIN WANG[1], JIALIN HONG[2], RUDOLF SCHERER[3] AND FENGSHAN BAI[4]

Abstract. Stochastic action integral and Lagrange formalism of stochastic Hamiltonian systems are written through construing the stochastic Hamiltonian systems as nonconservative systems with white noise as the nonconservative 'force'. Stochastic Hamilton's principle and its discrete version are derived. Based on these, a systematic approach of producing symplectic numerical methods for stochastic Hamiltonian systems, i.e., the stochastic variational integrators are established. Numerical tests show validity of this approach.

Key Words. Hamilton's principle, stochastic Hamiltonian systems, symplectic methods, variational integrators.

1. Introduction

The Hamiltonian formalism of a deterministic mechanical system is

$$dp = -\frac{\partial H}{\partial q}dt, \quad p(0) = p_0, \tag{1}$$

$$dq = \frac{\partial H}{\partial p}dt, \quad q(0) = q_0, \tag{2}$$

where $H(p, q)$ is Hamiltonian function. A stochastic Hamiltonian system is a Hamiltonian system under certain random disturbances, represented as (Milstein et al., [18])

$$dp = -\frac{\partial H}{\partial q}dt - \sum_{k=1}^{m} \frac{\partial H_k}{\partial q} \circ dW_k(t), \quad p(0) = p_0, \tag{3}$$

$$dq = \frac{\partial H}{\partial p}dt + \sum_{k=1}^{m} \frac{\partial H_k}{\partial p} \circ dW_k(t), \quad q(0) = q_0, \tag{4}$$

where $W_k(t)$ $(k = 1, \cdots, m)$ are m independent standard Wiener processes, called noises. The small circle '$\circ$' before $dW_k(t)$ denotes stochastic differential equations of Stratonovich sense.

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Both the deterministic and stochastic Hamiltonian systems have an intrinsic property-the symplecticity, i.e., the preservation of the symplectic structure (Hairer et al., [7], Milstein et al., [18], [19], Poincaré, [21])

$$dp(t) \wedge dq(t) = dp_0 \wedge dq_0, \quad \forall t \geq 0. \tag{5}$$

Geometrically, it means the preservation of area along phase flow of the system (Hairer, [7]).

In numerical simulation, property (5) of the theoretical solution $(p(t), q(t))$ is expected to be preserved by the numerical solution (p_n, q_n), that is

$$dp_{n+1} \wedge dq_{n+1} = dp_n \wedge dq_n, \quad \forall n \geq 1. \tag{6}$$

Such numerical methods are called symplectic methods. Since the qualitative behavior (5) is preserved, symplectic methods show significant superiority than non-symplectic methods, especially in long-time simulation. Pioneering work on deterministic symplectic methods goes back to de Vogelaere ([28] 1956), Ruth ([23] 1983) and Feng Kang et al. ([4] 1985, [5] 1986, [6] 1989). Since then, there has been an accelerating interest and effort on the study of such methods, which is now an important subject of computational mathematics and scientific computing. On the contrary, although there has been much effort on numerical methods for SDEs, e.g. [1], [2], [10], [11], [12], [13] etc., systematic research on stochastic symplectic methods, marked by the work of Milstein et al. ([18], [19], 2002), is still rare. In these works, they gave some symplectic Runge-Kutta type methods. Systematic construction of stochastic symplectic methods is still an open problem.

Variational integrators ([7], [14], [17], [25], [29]) have been an important approach of creating symplectic methods. They are tightly connected with the Hamilton's principle and its discrete version ([15], [16], [27]). For stochastic Hamiltonian systems, however, the main difficulty in constructing the variational integrators is the formulation of the stochastic Hamilton's principle.

In this article, we start from the point of view of construing the stochastic Hamiltonian systems as nonconservative systems, for which the white noise is a nonconservative 'force'. We then propose the formulation of the stochastic action integral, Euler-Lagrange equations of motion, as well as the stochastic Hamilton's principle. Based on these, the theory of stochastic variational integrators is constructed.

The second section derives the stochastic Hamilton's principle. Stochastic variational integrators are constructed in section 3. Section 4 are examples and numerical experiments.

2. Stochastic Hamilton's Principle

The Hamiltonian system (1)-(2) is a conservative mechanical system. Its Lagrangian formalism is ([7])

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}, \tag{7}$$

where $p = \frac{\partial L}{\partial \dot{q}}$, called the Legendre transform, and $L(q(t), \dot{q}(t))$ is the Lagrangian function.