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A ROBUST OVERLAPPING SCHWARZ METHOD FOR A SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM WITH MULTIPLE SOLUTIONS

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Abstract. An overlapping Schwarz domain decomposition is applied to a semilinear reaction-diffusion two-point boundary value problem with multiple solutions. Its diffusion parameter ε^2 is arbitrarily small, which induces boundary layers. The Schwarz method invokes two boundary-layer subdomains and an interior subdomain, the narrow overlapping regions being of width $O(\varepsilon|\ln\varepsilon|)$. Constructing sub- and super-solutions, we prove existence and investigate the accuracy of discrete solutions in particular subdomains. It is shown that when $\varepsilon \le CN^{-1}$ and layer-adapted meshes of Bakhvalov and Shishkin types are used, one iteration is sufficient to get second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the maximum norm uniformly in ε, where N is the number of mesh intervals in each subdomain. Numerical results are presented to support our theoretical conclusions.

Key Words. semilinear reaction-diffusion, singularly perturbed, boundary layers, domain decomposition, overlapping Schwarz method.

1. Introduction

Consider the singularly perturbed semilinear reaction-diffusion boundary value problem

$$Fu := -\varepsilon^2 u''(x) + f(x,u) = 0, \qquad x \in \Omega = (0,1), \tag{1a}$$

$$u(0) = g_0, \qquad u(1) = g_1, \tag{1b}$$

where ε is a small positive parameter, f is a sufficiently smooth function, and g_0 and g_1 are given constants. This is a one-dimensional version of the multidimensional reaction-diffusion equation $-\varepsilon^2\triangle u + f(x,u) = 0$, which we will consider in a future paper [10] (when posed in a smooth two-dimensional domain).

We shall examine solutions of (1) that exhibit sharp boundary layers, which are narrow regions where solutions change rapidly. In general, solutions of (1) may also have interior transition layers [12]. To obtain reliable numerical approximations of layer solutions in an efficient way, one has to use locally refined meshes that are fine and anisotropic in layer regions and standard outside. When multidimensional meshes of different nature are introduced in different subdomains, it might be rather inconvenient to match them; see, e.g., [7] for non-matching meshes used to solve

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a two-dimensional problem of type (1). Furthermore, different discretizations of differential equations might be used in layer regions and outside, in which case they should be matched along the interface boundaries; see, e.g., [9].

Handling non-matching meshes and matching different discretizations along the interface boundaries can be entirely avoided by invoking iterative overlapping domain decomposition methods of Schwarz-Chimera type; see, e.g., [18, §1.5]. Note that non-overlapping domain decomposition methods, at best, have conventional geometric rates of convergence when applied to singularly perturbed problems of type (1). In contrast, overlapping methods, with the overlapping regions being as narrow as $O(\varepsilon|\ln\varepsilon|)$, might enjoy much faster convergence. To be more precise, we prove in this paper that one iteration is sufficient to achieve second-order accurate computed solutions when $\varepsilon \le CN^{-1}$, where N is the number of mesh intervals in each subdomain; see Theorem 5.5 for details.

When considering semilinear problems of type (1), it is frequently assumed in the numerical analysis literature that $f_u(x,u) > \gamma^2 > 0$ for all $(x,u) \in \Omega\times\mathbb{R}$ and some positive constant γ. Under this assumption, our problem (1) and the associated reduced problem

$$f(x,u_0(x)) = 0 \qquad \text{for all} \quad x\in\Omega, \tag{2}$$

defined by setting $\varepsilon = 0$ in (1), have unique solutions u and u_0. This global assumption is however rather restrictive. E.g., mathematical models of biological and chemical processes frequently involve problems related to (1) with $f(x,u)$ that is non-monotone with respect to u. Therefore, we examine problem (1) under the following weaker assumptions also used in [4, 6, 11, 17, 19, 20]:

- it has a *stable reduced solution*, i.e. there exists a sufficiently smooth solution u_0 of (2) such that

$$f_u(x,u_0(x)) > \gamma^2 > 0 \qquad \text{for all} \;\; x\in\Omega; \tag{3a}$$

- the boundary data g_l, for $l = 0,1$, satisfy

$$\int_{u_0(l)}^{v} f(l,s)\,ds > 0 \qquad \text{for all} \;\; v \in (u_0(l),\, g_l\,]'. \tag{3b}$$

  Here the notation $(a,b]'$ is defined to be $(a,b]$ when $a<b$ and $[b,a)$ when $a>b$, while $(a,b]' = \emptyset$ when $a=b$.

Conditions (3) intrinsically arise from the asymptotic analysis of problem (1) and guarantee that there exists a boundary-layer solution u such that $u\approx u_0$ in the interior part of Ω, while the boundary layers are of width $O(\varepsilon|\ln\varepsilon|)$; see, e.g., [6, 17, 20]. Note that assumption (3a) is local, i.e. the reduced problem (2) is permitted to have more than one stable solution. Furthermore, if multiple stable solutions of the reduced problem satisfy (3b), then problem (1) has multiple boundary-layer solutions.

We shall now present a continuous version of the discrete Schwarz method that we investigate in this paper. Consider the overlapping subdomains

$$\Omega_{\mathrm{L}} = (0,2\sigma), \qquad \Omega_{\mathrm{C}} = (\sigma,1-\sigma), \qquad \Omega_{\mathrm{R}} = (1-2\sigma,1), \tag{4}$$

where $\sigma\in(0,\,1/4]$ is a parameter, which throughout the paper will satisfy $\sigma \ge (2/\gamma)\,\varepsilon\ln N$. Let u_{L}, u_{R}, and then u_{C} be solutions of the following boundary value